An Improved Upper Bound on the Crossing Number of the Hypercube*

Luerbio Faria,[1] Celina Miraglia Herrera de Figueiredo,[2] Ondrej Sýkora,[3] and Imrich Vrt'o[4]

[1] *FACULDADE DE FORMAÇÃO DE PROFESSORES, DEPARTAMENTO DE MATEMÁTICA*
UNIVERSIDADE DO ESTADO DO RIO DE JANEIRO
SÃO GONÇALO, RJ, BRAZIL
E-mail: luerbio@cos.ufrj.br

[2] *PROGRAMA DE ENGENHARIA DE SISTEMAS E COMPUTAÇÃO, COPPE*
UNIVERSIDADE FEDERAL DO RIO DE JANEIRO
CAIXA POSTAL 68511
21941-972 RIO DE JANEIRO, RJ, BRAZIL
E-mail: celina@cos.ufrj.br

[3] *DEPARTMENT OF COMPUTER SCIENCE*
LOUGHBOROUGH UNIVERSITY
LOUGHBOROUGH, LEICESTERSHIRE LE11 3TU, UNITED KINGDOM

[4] *DEPARTMENT OF INFORMATICS*
INSTITUTE OF MATHEMATICS
SLOVAK ACADEMY OF SCIENCES
DÚBRAVSKÁ 9, 841 04 BRATISLAVA, SLOVAK REPUBLIC
E-mail: vrto@savba.sk

*An extended abstract of this article was presented at WG'2003, Graph Theoretical Concepts in Computer Science and appeared in *Lecture Notes in Computer Science* **2880** (2003), 230–236.

Contract grant sponsor: CNPq; Contract grant sponsor: CAPES; Contract grant sponsor: FAPERJ; Contract grant sponsor: FINEP, Brazilian Research Agencies; Contract grant sponsor: VEGA; Contract grant number: 2/0111/09 ; Contract grant number: APVV-0433/06.

Received May 31, 2007; Revised April 18, 2008

Published online 18 June 2008 in Wiley InterScience(www.interscience.wiley.com).
DOI 10.1002/jgt.20330

Abstract: We draw the n-dimensional hypercube in the plane with $\frac{5}{32}4^n - \lfloor\frac{n^2+1}{2}\rfloor 2^{n-2}$ crossings, which improves the previous best estimation and coincides with the long conjectured upper bound of Erdös and Guy. J Graph Theory 59: 145–161, 2008

Keywords: *topological graph theory; crossing number; n-cube*

1. INTRODUCTION

The crossing number of a graph G, denoted by $\mathrm{cr}(G)$ is the minimum number of crossings of its edges among all drawings of G in the plane. It is a fundamental topological invariant but appears naturally in the design of VLSI circuits [8] and visualization of graph like structures [1]. The problem is NP-hard [5]. There are only a few infinite families of graphs for which exact crossing numbers are known. See, for example, surveys [9,11]. One of the most challenging problems is the crossing number of the hypercube graph. The hypercube Q_n is defined in the standard way. The vertices are all binary strings of length n and two vertices are adjacent iff the corresponding strings differ in one position. In 1969, Harary [7] mentioned that there does not even exist a conjecture about the crossing number of the hypercube. Then, Eggleton and Guy [2] announced a drawing which implies that for $n \geq 3$

$$\mathrm{cr}(Q_n) \leq \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}. \tag{1}$$

Later a gap was found in the construction [6]. However, Erdös and Guy [3] conjectured equality in (1). Madej [10] proposed a drawing of Q_n with

$$\frac{1}{6}4^n - n^2 2^{n-3} - 3\cdot 2^{n-4} + \frac{1}{48}(-2)^n$$

crossings and showed that $\mathrm{cr}(Q_5) \leq 56$. Sýkora and Vrt'o [12] proved that Madej's bound is asymptotically optimal by deriving the following lower bound:

$$\mathrm{cr}(Q_n) \geq \frac{1}{20}4^n + O(n^2 2^n).$$

Then Dean and Richter showed that $\mathrm{cr}(Q_4) = 16$, which is the only exact result in this area apart from the trivial observation $\mathrm{cr}(Q_3) = 0$. Recently, Faria and de

Figueiredo [4] decreased the Madej's upper bound to

$$\frac{165}{1024}4^n - (2n^2 - 11n + 34)2^{n-3}, \tag{2}$$

which coincides with the RHS of (1) up to $n \le 8$.

In this article we construct a new drawing of Q_n in the plane which has the conjectured number of crossings

$$\frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}.$$

2. PRELIMINARIES

In this section we consider some topological results necessary to establish the counting of crossings of our proposed family of drawings. For this purpose, we define next four structures: M_1^i, M_2^i, M_3^i, and M_1^ic, called "meshes" which are used in the counting process of the number of crossings. Specifically, only M_3^i and M_1^ic are used to evaluate the number of crossings, M_1^i and M_2^i are auxiliary to the construction of M_3^i and M_1^ic.

We consider the canonical geometry of the real plane $\mathbb{R}^2$. We denote by [0, 1] the closed interval joining the points (0, 0) and (1, 0) of the horizontal real axis. Let n be a positive integer. Let r and s be a non-horizontal pair of parallel straight lines in the real plane $\mathbb{R}^2$, such that the point (0, 0) belongs to r and the point (1, 0) belongs to s. Let $S = \{(r_i, s_i) : i \in \{1, 2, 3, \ldots n\}\}$ be a set of non-horizontal pairs of parallel straight lines in the real plane $\mathbb{R}^2$, such that the point (0, 0) belongs to r_i and the point (1, 0) belongs to s_i.

We call *mesh one* index n (M_1^n) the set of points of the plane consisting of the points of the n-element set S plus the points in the interval [0, 1]. In Figure 1 we show as an example a drawing of each M_1^1, M_1^2, M_1^3, and M_1^5.

In Lemma 1, we evaluate the number of crossings of M_1^n.

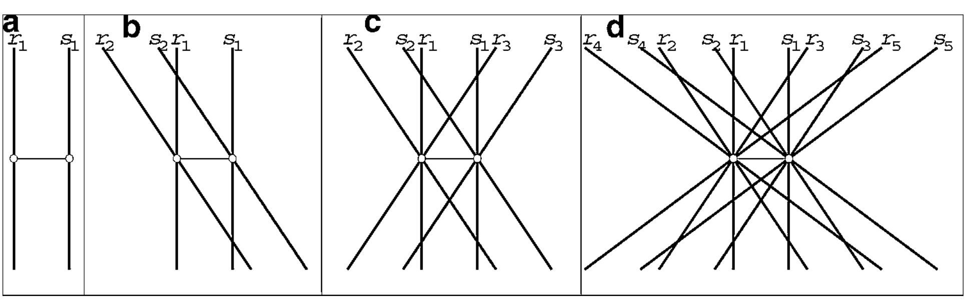


FIGURE 1. Drawings of (**a**) M_1^1, (**b**) M_1^2, (**c**) M_1^3, and (**d**) M_1^5.

Lemma 1. *Let n be a positive integer, then there is a drawing of M_1^n with*

$$i(n) = n(n-1) \text{ crossings.}$$

Proof. We argue by induction. Lemma 1 is true for $n = 1$. Let (r, s) be the additional pair of parallel straight lines we add to M_1^{n-1}. Then there are 2 additional crossings which are yielded between (r, s) and each one of $n-1$ pairs of parallel straight lines of M_1^{n-1}, that is, we have $2(n-1)$ additional crossings. Suppose that for each $k < n$ there is a drawing of M_1^k with $i(k) = k(k-1)$ crossings. Hence, we obtain a drawing of M_1^n with $i(n) = n(n-1) = (n-1)(n-2) + 2(n-1)$ crossings from a drawing of M_1^{n-1}, since the drawing of M_1^{n-1} has $i(n-1) = (n-1)(n-2)$ crossings. ■

Let $S = \{(r_i, s_i) : i \in \{1, 2, 3, \ldots n\}\}$ be the set of non horizontal pairs of parallel straight lines corresponding to the set of straight lines of a drawing of M_1^n with $n(n-1)$ crossings. Consider R the innermost region of the upper semi-plane containing the interval $[0, 1]$ bounded by the elements of S. Let $I = [P, Q]$ be a horizontal closed interval of the plane contained in the interior of R. For each $i \in \{1, 2, 3, \ldots n\}$ consider t_i and u_i the pair of parallel straight lines each parallel to r_i, respectively, containing the points P and Q.

The *mesh two* index n (M_2^n) consists of a drawing of the set S of parallel straight lines $S = \{r_i, s_i, t_i, u_i : i \in \{1, 2, 3, \ldots n\}\}$ and of the points of the trapezium defined by the points $(0, 0)$, P, Q and $(1, 0)$. In Figure 2, we show as an example a drawing of each M_2^1, M_2^2, M_2^3, and M_2^5.

In Lemma 2, we evaluate the number of crossings of M_2^n.

Lemma 2. *Let n be a positive integer, then there is a drawing of M_2^n with*

$$j(n) = 2[i(n) + n(2n-1)] = 2n[n - 1 + (2n-1)] = 2n(3n-2) \text{ crossings.}$$

Proof. Let $S = \{r_i, s_i, t_i, u_i : i \in \{1, 2, 3, \ldots n\}\}$ and the points $(0, 0)$, P, Q and $(1, 0)$ be, respectively, the set of parallel straight lines and the points of the trapezium of a corresponding drawing of M_2^n. First of all, we note that in a drawing of M_2^n we have twice the crossings of M_1^n, once given by the crossings among

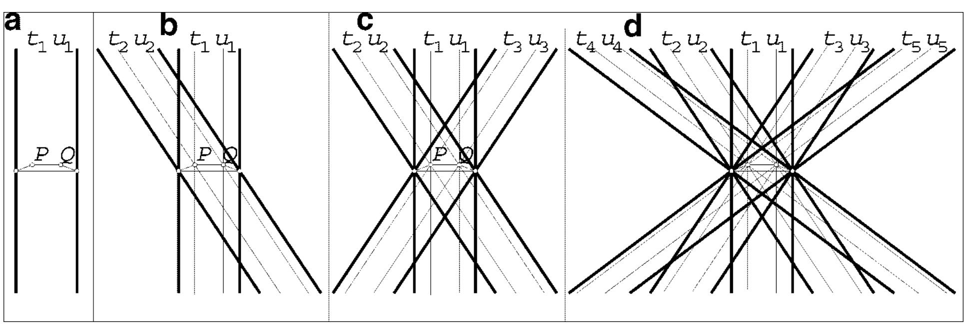


FIGURE 2. Drawings of (**a**) M_2^1, (**b**) M_2^2, (**c**) M_2^3, and (**d**) M_2^5.

the parallel straight lines of the set $S_1 = \{r_i, s_i, : i \in \{1, 2, 3, \dots n\}\}$, and once given by the crossings among the parallel straight lines of the set $S_2 = \{t_i, u_i : i \in \{1, 2, 3, \dots n\}\}$.

We observe that for each $i \in \{1, 2, 3, \dots n\}$ the straight line t_i:

(1) has exactly one crossing with one straight line in the pair of parallel straight lines r_j, s_j, where $j \in \{1, 2, 3, \dots n\} \setminus \{i\}$ in the upper semi-plane;
(2) has exactly one crossing with one straight line in the pair of parallel straight lines r_j, s_j, where $j \in \{1, 2, 3, \dots n\} \setminus \{i\}$ in the lower semi-plane;
(3) has exactly one crossing with the interval $[0, 1]$.

Hence, we have that the number of crossings between the straight lines of S_1 and the straight line t_i is $(n-1)+(n-1)+1 = 2n-1$ crossings. Analogously, we have the number $2n-1$ of crossings between the straight lines of S_1 and the straight line u_i. Hence, we have $2n(2n-1)$ crossings between the straight lines of S_1 and the straight lines of S_2.

Thus, there is a drawing of M_2^n with $j(n) = 2[i(n) + n(2n-1)] = 2n[n-1+(2n-1)] = 2n(3n-2)$ crossings. ■

The *mesh three* index n (M_3^n) consists of a M_2^n plus a suitable modification, where we remove one semi-straight line and add five additional curves defined next:

(1) Let q be the leftmost semi-straight line of M_2^n starting in P which crosses the lower semi-plane. Remove q from M_2^n.
(2) We add to M_2^n, the curve q' which starts in P, contains only the point P of the trapezium, is asymptotical to the leftmost semi-straight line containing $(0, 0)$ crossing the lower semi-plane; and crosses the other straight lines containing P only in the point P.
(3) The curves r', s', t' and u' defined below do not intersect each other.
(4) We add to M_2^n, two horizontal semi-straight lines r' and s', respectively, starting in the vertices Q and $(1, 0)$ of the trapezium, each of the semi-straight lines r' and s' takes, with respect to the x-axis, the direction of $+\infty$.
(5) We add to M_2^n, the curve t' which starts in P, contains only the point P of the trapezium, is asymptotical to the semi-straight line r', and crosses the other straight lines containing P only in the point P.
(6) We add to M_2^n, the curve u' which starts in $(0, 0)$, contains only the point $(0, 0)$ of the trapezium, is asymptotical to the semi-straight line s', and crosses the other straight lines containing $(0, 0)$ only in the point $(0, 0)$.

For the convenience of the reader, we offer in Figure 3 an example with a drawing of each M_3^1, M_3^2, M_3^3, and M_3^5.

In Lemma 3, we evaluate the number of crossings of M_3^n.

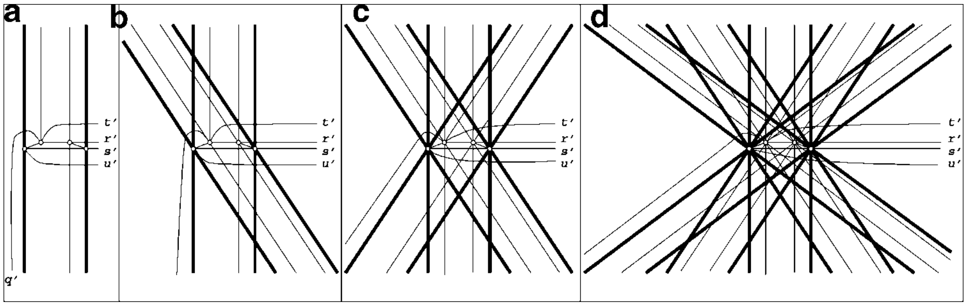


FIGURE 3. Drawings of (**a**) M_3^1, (**b**) M_3^2, (**c**) M_3^3, and (**d**) M_3^5.

Lemma 3. *The number of crossings in* M_3^n *is*

$$k(n) = j(n) + 3n + 3n - 1 = j(n) + 6n - 1 = 2n(3n - 2) + 6n - 1$$
$$= 6n^2 + 2n - 1.$$

Proof. We count the number of crossings in M_3^n by deriving the resulting number of crossings when we remove and add curves in order to obtain M_3^n from M_2^n. By Lemma 2 the number of crossings of M_2^n is $j(n)$ crossings.

The semi-straight line q has n crossings in M_2^n. The added curve q', it crosses the same $n - 1$ straight lines as q except the crossing with the interval [0, 1]. In addition, q' has a crossing with one of the straight lines parallel to q. Hence, there are n crossings in q'. Thus, the removal of q and the addition of q' do not modify, $j(n)$, the number of crossings in M_2^n.

The semi-straight line s' has no crossing. The semi-straight line r' crosses each one of the n semi-straight lines which crosses the upper semi-plane and starts in (1, 0). The curve t' crosses each one of the n semi-straight lines that crosses the upper semi-plane and starts in Q and each one of the n semi-straight lines that crosses the upper semi-plane and starts in (1, 0). Hence, there are $2n$ crossings in t'. The curve u' crosses each one of the n semi-straight lines that crosses the lower semi-plane and starts in Q and each one of the n semi straight lines that crosses the upper semi-plane and starts in (1, 0) and the $n - 1$ semi-straight lines which starts in P. Here we remark that the straight line q is replaced by curve q' which has no crossing with the curve u'. Hence, there are $3n - 1$ crossings in u'. Altogether we have the number of crossings $k(n) = j(n) + 3n + 3n - 1 = j(n) + 6n - 1 = 2n(3n - 2) + 6n - 1 = 6n^2 + 2n - 1$. ■

We call *chopped mesh* 1 index n ($M_1^n c$) the set of points of M_1^n without a pair of parallel semi-straight lines of the left-most lower semi-plane. In Figure 4, we show a drawing of $M_1^1 c$, $M_1^2 c$, $M_1^3 c$, and $M_1^5 c$.

Lemma 4. *Let* n *be a positive integer. There is a drawing* $M_1^n c$ *with* $(n - 1)^2$ *crossings.*

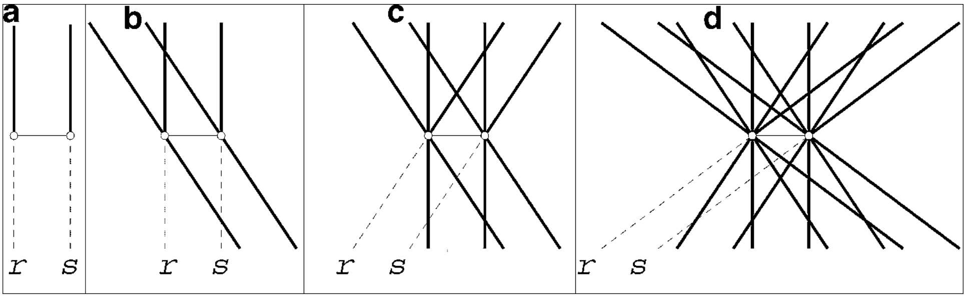


FIGURE 4. Drawings of (**a**) M_1^1c, (**b**) M_1^2c, (**c**) M_1^3c, and (**d**) M_1^5c. Dashed straight lines represent the straight lines r and s which are removed in order to define the corresponding drawing to M_1^nc.

Proof. By Lemma 1 we have that there is a drawing of M_1^n with $n(n-1)$ crossings. We obtain a drawing of M_1^nc by removing r which is the leftmost semi-straight line in the lower semi-plane and s its corresponding parallel semi-straight line in the lower semi-plane. Note that the semi-straight line r has no crossing while the semi-straight line s has $n-1$ crossings, one for each of the semi-straight lines starting in (0, 0) which cross the lower semi-plane. Hence we have $n(n-1)-(n-1)=(n-1)^2$ crossings in our drawing of M_1^nc. ■

3. THE PROPOSED FAMILY OF DRAWINGS FOR Q_n

Recall the definition of the hypercube. We say that an edge belongs to the ith dimension if its end-vertices (strings) differ in the ith position from the left. A vertex is called an *even* vertex if the number of 1's in its corresponding string is even. Let C_m denote an m-vertex cycle. For graphs G and H, let $G \square H$ denote their Cartesian product.

We consider n odd and n even separately. For every positive integer k we describe a drawing for Q_n, where $n=2k-1$. Second, we use this construction in order to establish the corresponding construction for $n+1$. Given a drawing D of a graph G, we denote by $cr(D)$ the *number of crossings of the drawing D*.

A. Odd Case

Let $n=2k-1$, for k an odd positive integer. The main idea of the construction is the following. Assume we have a suitable drawing D_n of Q_n in the plane with the claimed number of crossings, denoted by $\mathrm{cr}(D_n)$. We utilize the identity $Q_{n+2} = Q_n \square C_4$. Replace every vertex v of Q_n in the "small" neighborhood of v in the drawing D_n by a 4-cycle. Now every drawn edge e in D_n which started in v will be replaced by 4 "parallel" edges (*bunch*) and drawn along the original edge e. Notice that, in this case, locally we have a drawing of mesh M_3^{k-1}. Doing this carefully we get a drawing of Q_{n+2}. The total number of crossings will be the number of the crossings in the small neighborhood of the 4-cycle times 2^n plus $16\mathrm{cr}(D_n)$.

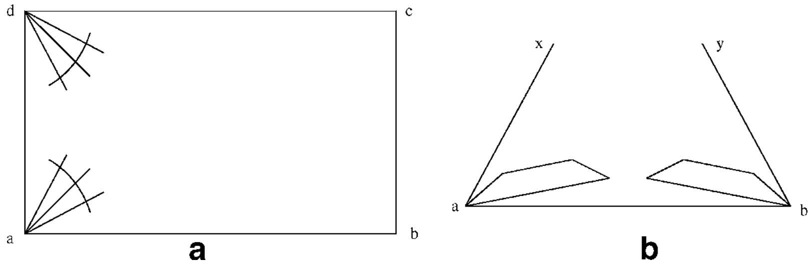


FIGURE 5. The number of edges between edges *ab* and *ad* is equal to the number of edges between edges *dc* and *da* in Figure (a). Two 4-cycles drawn between edge *ab* and the first counterclockwise edge *ax*, and between edge *ab* and the first clockwise edge *by* in Figure (b).

We construct inductively a drawing D_n of Q_n satisfying the following five properties:

Property 1. *The number of crossings in the drawing is:*

$$\mathrm{cr}(D_n) = \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}.$$

Property 2. *Edges of the same dimension do not cross.*

Property 3. *In any* 4*-cycle abcd, where ab*, *cd are of the nth dimension and ad*, *bc are of the ith dimension,* $i < n$*, it holds that the number of edges drawn between edges ab and ad in the counterclockwise order, and the number of edges drawn between dc and da in the clockwise order are the same. See Figure* 5*a.*

Property 4. *Take any edge ab of the nth dimension, where a is an even vertex. In a small neighborhood of a, a* 4*-cycle is attached to a as in Figure* 5*b, that is, the* 4*-cycles do not interfere with the current drawing, and the* 4*-cycle is placed between the edge ab and its neighbor ax in the counterclockwise order.*

Draw a symmetrical 4-cycle attached at the vertex b lying between the edge ab and its neighbor by in the clockwise order. Call these 4-cycles *new* 4-cycles. Consider again any 4-cycle $abcd$, where ab, cd are edges of the nth dimension, ad, bc are of the ith dimension, $i < n$, and a is an even vertex. Starting with the edge ab let the edge of the ith dimension be the $r(i)$th in the clockwise direction around a vertex a.

(1) If $r(i) > k$ then the new four 4-cycles attached to a, b, c, d lie in the bounded region of the cycle $abcd$ as one can see in Figure 6a.
(2) If $r(i) \le k$ then the new 4-cycles attached to a, b, c, d lie in the unbounded region of the cycle $abcd$ as one can see in Figure 6b.

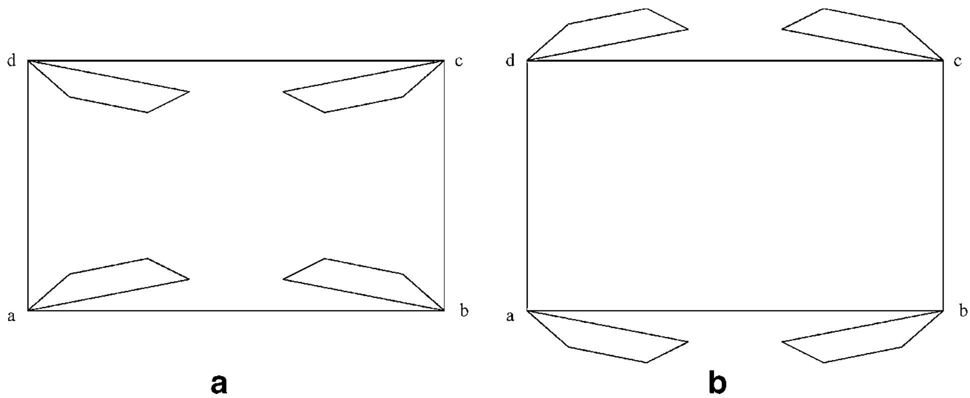


FIGURE 6. The drawing of the new 4-cycles according to *r(i)* $>$ *k* in Figure (a), or according to *r(i)* $\leq$ *k* in Figure (b).

Property 5. *Consider the five types of bunches of four edges from a* 4*-cycle, corresponding to a mesh* M_3^n*, spanned by edges of dimensions* n *and* $n-1$*, depicted in Figure* 7*. Whenever* $k \geq 3$, $n = 2k-1 \geq 5$*, then a bunch of four edges of same type is joint to a bunch of edges of same type. In Figure* 8*, we have depicted the five join types. In Figure* 9*, we show that when the meshes replace the vertices we produce the same collection of join types.*

This concludes the inductive construction for odd n.

Let $n = 3$. Then Figure 10 shows a drawing of Q_3 satisfying the above five properties.

Assume we have a drawing of Q_n, for some $n \geq 3$ satisfying the five properties. In what follows we construct a drawing of Q_{n+2} satisfying the five properties.

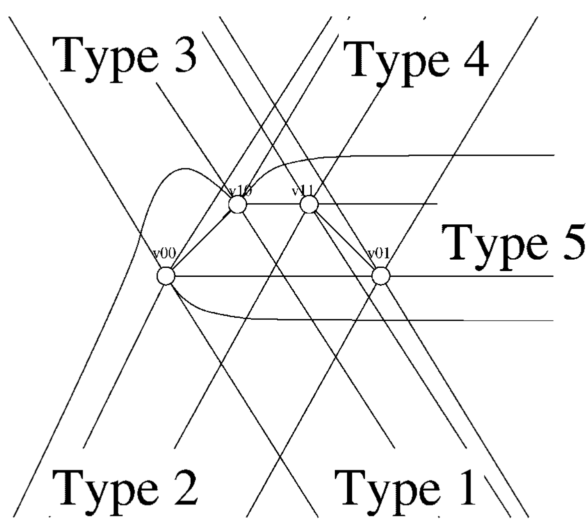


FIGURE 7. Bunch types.

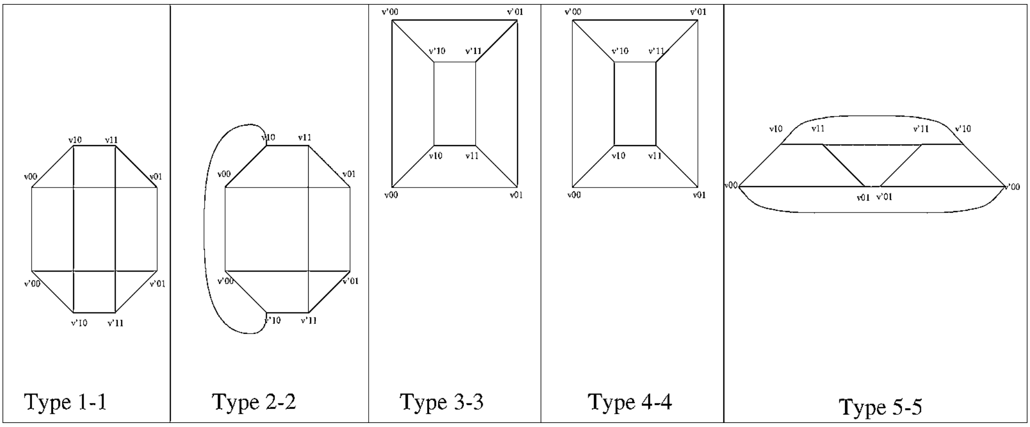


FIGURE 8. Definition for the kind of joins of the same types.

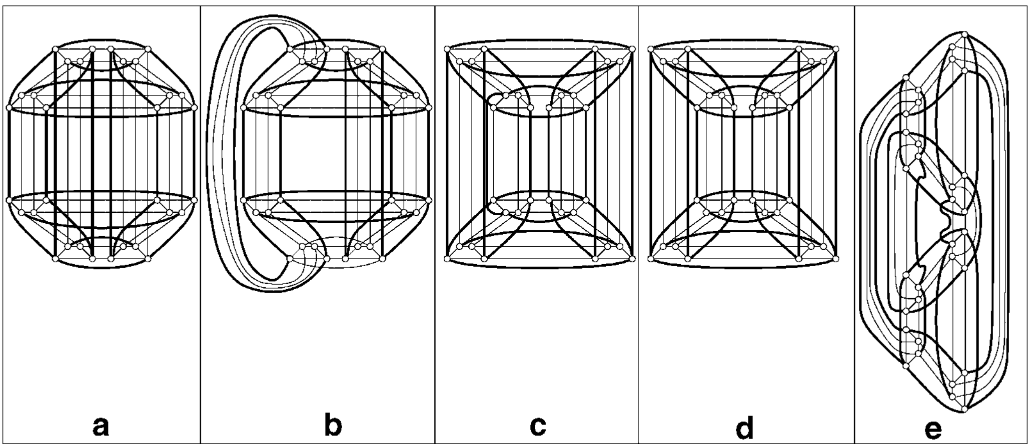


FIGURE 9. **a**: Join types 1-1, 3-3 or 4-4 and 5-5 in Q_{n+2} obtained from a join type 1-1 in Q_n. **b**: Join types 1-1, 3-3 or 4-4 and 5-5 in Q_{n+2} obtained from a join type 2-2 in Q_n. **c**: Join types 1-1, 2-2, 3-3 or 4-4 and 5-5 in Q_{n+2} obtained from a join type 3-3 in Q_n. **d**: Join types 1-1, 3-3 or 4-4 and 5-5 in Q_{n+2} obtained from a join type 4-4 in Q_n. **e**: Join types 1-1, 2-2, 3-3 or 4-4 and 5-5 in Q_{n+2} obtained from a join type 5-5 in Q_n.

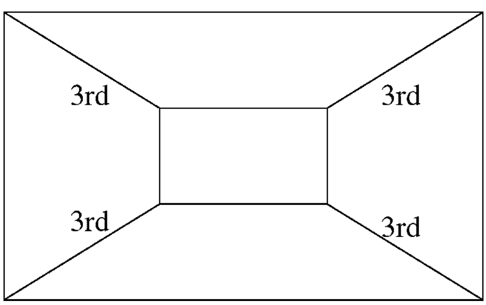


FIGURE 10. Q_3.

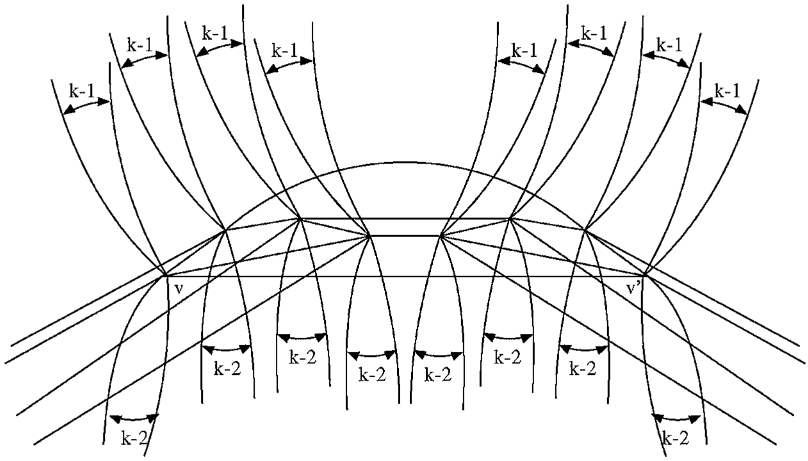


FIGURE 11. Valid patterns for the edges emanating from the vertices of the two 4-cycles in the clockwise direction: upward $k-1$ bunches of edges, horizontal bunch of edges, and downward bunches of edges.

Take any even vertex v in the drawing of Q_n. Let v' be its neighbor along the nth dimension. Without producing new crossings, draw two new 4-cycles on vertices $v00 = v$, $v01$, $v11$, $v10$ and $v'00 = v'$, $v'01$, $v'11$, $v'10$ such that the edge $v00$, $v01$ ($v'00$, $v'01$) is a "neighbor" of the edge vv', in the drawing. The edge pattern in the original vertices v and v' is repeated in all vertices of the cycle corresponding to v and v', respectively. And the new edges are routed as depicted in Figure 11.

We complete the drawing of Q_{n+2} by describing the routing of the new edges. The drawing of the edges of the ith dimension, $i = n, n+1, n+2$, is obvious from Figure 11. Let $i < n$. Consider any four cycle v, v', u, u' in the drawing of Q_n, where the edges vv' and uu' belong to the nth dimension, and the edges vu' and $v'u$ belong to the ith dimension, and v is an even vertex. Starting with the edge vv', let the edge of the ith dimension be the $r(i)$th in the clockwise direction around the vertex v. Consider now the corresponding cycle $v00$, $v'00$, $u00$, $u'00$ in the "partial" drawing of Q_{n+2}.

Distinguish 3 cases (see Fig. 12):

(1) Let $r(i) > k$. Note that in this case the new 4-cycles lie in the bounded region of the 4-cycle $vv'uu'$. Draw the connection between vjl and $u'jl$ for $j, l, = 0, 1$, $j + l > 0$ parallel with the "old" edge $v00$ and $u'00$ using the valid join type for one of the $k-1$ upward bunches of edges (Figs. 11 and 12 case $r(i) > k$).

(2) Let $r(i) = k$. In this case the new 4-cycles lie in the unbounded region of the 4-cycle $vv'uu'$. Use the valid join type for the horizontal bunch of edges (Figs. 11 and 12 case $r(i) = k$).

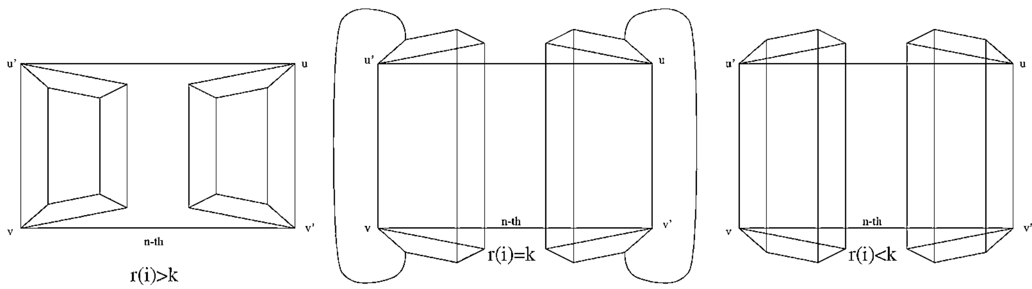


FIGURE 12. The routing of the bunches according to the edge $r(i)$th dimension.

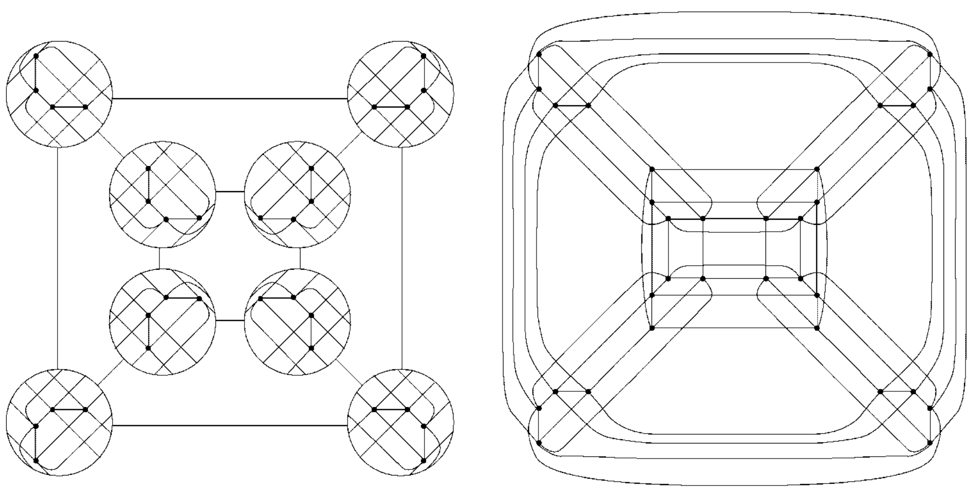
(3) Let $r(i) < k$. In this case again the new 4-cycles lie outside the 4-cycle $vv'uu'$. Use the valid join type for one of the $k-2$ downward bunches of edges (Figs. 11 and 12 case $r(i) < k$).

We make a similar drawing for the 4-cycles attached to $v'00$ and $u00$.

In Lemma 6, the main lemma of the article, we prove that if n is odd and Property 1 holds for the drawing D_n of Q_n, then Property 1 holds for the drawing D_{n+2} of Q_{n+2}. Here we show that the drawing D_{n+2} satisfies properties 2–5. Property 2 is obviously fulfilled. Properties 3 and 4 can be checked by taking the 4-cycle $v00$, $v01$, $u'01$, $u'00$, formed by edges of the $(n+2)$nd dimension and the ith dimension, $i < n+2$ and using the inductive assumption. Property 5 can be checked in Figures 8 and 9. The drawing immediately implies that the produced graph is Q_{n+2}.

The proposed drawing of Q_{n+2} is concluded.

For the convenience of the reader we offer in Figures 13 and 14 a drawing for Q_5 and Q_7 obtained according to the rules of the construction.



a **b**

FIGURE 13. **a**: Drawing D_3 of Q_3 plus a 4-cycle for each vertex, indicating the routing of the bunches for each edge and (**b**) drawing D_5 of Q_5 obtained from D_3.

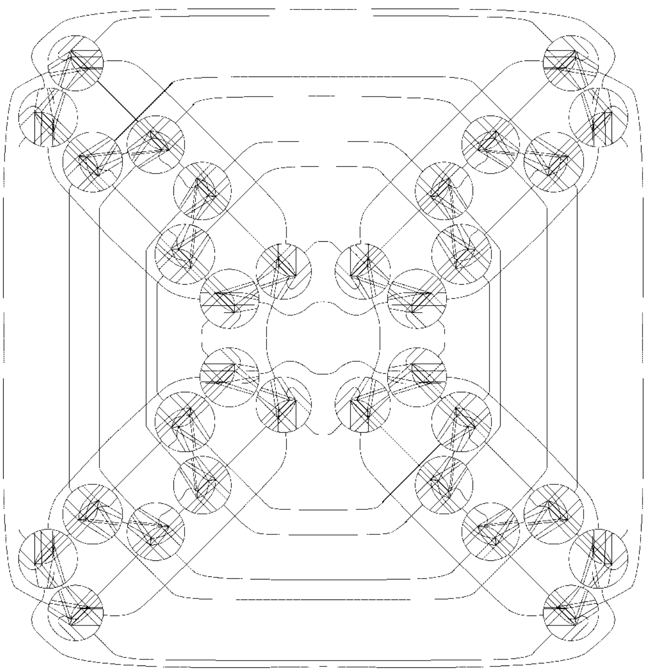


FIGURE 14. Indication of the routing for the bunches of four edges of the drawing D_7 of Q_7 obtained from the drawing D_5 of Q_5.

B. Even Case

We start with the drawing D_n of Q_n described above, for n odd. We utilize the identity $Q_{n+1} = Q_n \,\square\, P_2$, where P_2 denotes a 2 vertex path. The construction proceeds similarly as above but in a simpler way. To every vertex u in D_n, attach a new edge uv in a small neighborhood of u. For every edge e, starting in u, draw a new edge which starts in v and goes "parallel" to e. Notice that, in this case,

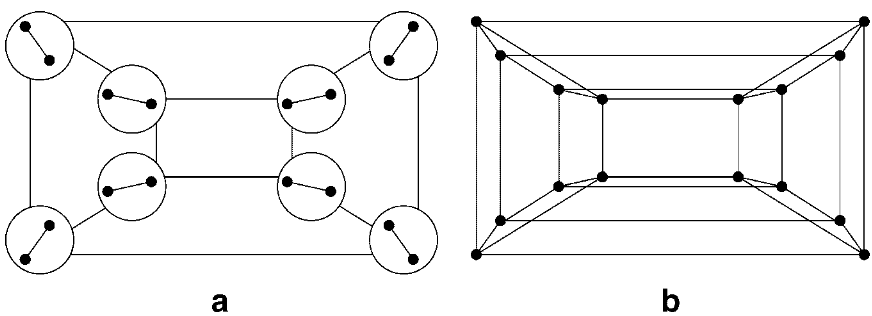


FIGURE 15. Drawing D_3 of Q_3 plus an edge for each vertex in (**a**) and drawing D_4 of Q_4, in (**b**), obtained from (a) by the addition of an edge for each suitable pair of vertices.

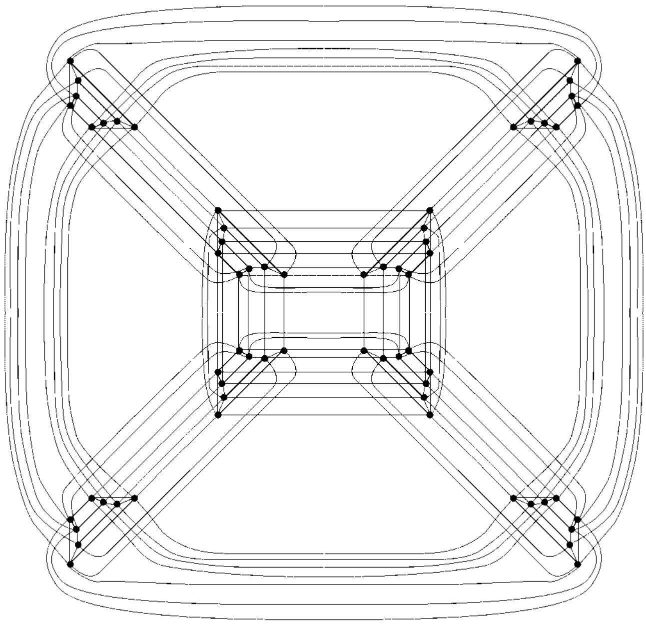


FIGURE 16. D_6.

locally we have a drawing of mesh $M_1^{\frac{n+1}{2}}c$. Doing this carefully we get a drawing of Q_{n+1}.

For the convenience of the reader we offer in Figures 15 and 16 drawings for Q_4 and Q_6 obtained according to the rules of construction.

4. THE CALCULATION OF THE NUMBER OF CROSSINGS IN THE PROPOSED FAMILY

Lemma 5. *If n is an odd positive integer, then the number of crossings produced "in the neighborhood" of the new* 4*-cycle attached in* $v00$ *is* $(3n^2 - 4n - 1)/2$.

Proof. We observe that the "neighborhood" of the vertex v_{00} in Figure 11 corresponds to a drawing of a M_3^{k-1}, whose number of crossings, by Lemma 3, is $6(k-1)^2 + 2(k-1) - 1 = 6k^2 - 10k + 3 = \frac{12k^2-20k+6}{2} = \frac{3(4k^2-4k+1)-4(2k-1)-1}{2} = \frac{3(2k-1)^2-4(2k-1)-1}{2} = \frac{3n^2-4n-1}{2}$ crossings. ■

Lemma 6. *If n is an odd positive integer, then*

$$cr(D_n) = \frac{5}{32}4^n - \left(\frac{n^2+1}{2}\right)2^{n-2}.$$

Proof. We argue by induction. The Lemma is valid if $n = 1$ or $n = 3$. We prove next that if n is an odd positive integer and the drawing D_n of Q_n

has $\frac{5}{32}4^n - (\frac{n^2+1}{2})2^{n-2}$ crossings, then the drawing D_{n+2} of Q_{n+2} has $\frac{5}{32}4^{n+2} - \lfloor\frac{(n+2)^2+1}{2}\rfloor 2^{(n+2)-2}$ crossings.

By Lemma 5 the total number of crossings in the drawing D_{n+2} of Q_{n+2} is

$$\begin{aligned}
\mathrm{cr}(D_{n+2}) &= 16\mathrm{cr}(D_n) + \frac{3n^2-4n-1}{2}2^n \\
&= 16\left(\frac{5}{32}4^n - \left(\frac{n^2+1}{2}\right)2^{n-2}\right) + (3n^2-4n-1)2^{n-1} \\
&= \frac{5}{32}4^{n+2} - \frac{4n^2+4}{2}2^{(n+2)-2} + (3n^2-4n-1)\frac{2^n}{2} \\
&= \frac{5}{32}4^{n+2} - \frac{4n^2+4-3n^2+4n+1}{2}2^n = \frac{5}{32}4^{n+2} - \frac{n^2+4n+4+1}{2}2^n \\
&= \frac{5}{32}4^{n+2} - \frac{(n+2)^2+1}{2}2^n = \frac{5}{32}4^{n+2} - \left\lfloor\frac{(n+2)^2+1}{2}\right\rfloor 2^{(n+2)-2},
\end{aligned}$$

and the inductive step for odd case is completed. ■

Lemma 7. *If n is an even positive integer, then the number of crossings produced "in the neighborhood" of the new edge in a drawing D_n of Q_n is $((n-2)/2)^2$.*

Proof. Recall a vertex of D_{n-1} has degree $n-1$. We yield D_n from D_{n-1} by adding a new edge attached to each vertex of D_{n-1} and the corresponding new edges defined in the construction for the even case. Hence, in the "neighborhood" of each new edge there is a drawing corresponding to a $M_1^{\frac{n}{2}}c$, which by definition has $\frac{2n}{2} - 1 = n-1$ pairs of semi-straight lines, totalizing degree n for each vertex of D_n. From Lemma 4, each $M_1^{\frac{n}{2}}c$ contributes with $\left(\frac{n}{2}-1\right)^2 = \left(\frac{n-2}{2}\right)^2$ additional crossings. ■

Lemma 8. *If n is an even positive integer, then*

$$cr(D_n) = \frac{5}{32}4^n - \left\lfloor\frac{n^2+1}{2}\right\rfloor 2^{n-2}.$$

Proof. Consider n an even positive integer. Next we prove that the drawing D_n of Q_n satisfies $\mathrm{cr}(D_n) = \frac{5}{32}4^n - \lfloor\frac{n^2+1}{2}\rfloor 2^{n-2}$ crossings.

By Lemma 6,

$$cr(D_{n-1}) = \frac{5}{32}4^{n-1} - \left(\frac{(n-1)^2+1}{2}\right)2^{(n-1)-2}.$$

By Lemma 7 the total number of crossings in the drawing D_n of Q_n is,

$$
\begin{aligned}
\mathrm{cr}(D_n) &= 4\mathrm{cr}(D_{n-1}) + \left(\frac{n-2}{2}\right)^2 2^{n-1} \\
&= 4\left(\frac{5}{32}4^{(n-1)} - \frac{(n-1)^2+1}{2}2^{(n-1)-2}\right) + \left(\frac{n-2}{2}\right)^2 2^{n-1} \\
&= \frac{5}{32}4^n - \frac{2n^2-4n+2+2}{2}2^{n-2} + \frac{n^2-4n+4}{2}2^{n-2} \\
&= \frac{5}{32}4^n - \frac{2n^2-4n+4-n^2+4n-4}{2}2^{n-2} = \frac{5}{32}4^n - \frac{n^2}{2}2^{n-2} \\
&= \frac{5}{32}4^n - \left\lfloor \frac{n^2+1}{2} \right\rfloor 2^{n-2}.
\end{aligned}
$$

■

Theorem 1. *For every positive integer, $cr(Q_n) \leq \frac{5}{32}4^n - \lfloor \frac{n^2+1}{2} \rfloor 2^{n-2}$.*

Proof. It follows from Lemmas 6 and 8. ■

ACKNOWLEDGMENTS

We thank the referees for their careful reading and valuable suggestions, which helped improve the presentation of this article. This work was partially supported by CNPq, CAPES, FAPERJ and FINEP, Brazilian research agencies and also partially supported by grants VEGA 2/0111/09 and APVV-0433/06.

REFERENCES

[1] G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis, Graph Drawing: Algorithms for Visualizations of Graphs, Prentice Hall, New York, NY, USA, 1999.

[2] R. B. Eggleton and R. P. Guy, The crossing number of the n-cube, Amer Math Soc Notices 17 (1970), 757.

[3] P. Erdös and R. K. Guy, Crossing number problems, Amer Math Monthly 80 (1973), 52–58.

[4] L. Faria and C. M. H. de Figueiredo, On Eggleton and Guy conjectured upper bound for the crossing number of the n-cube, Mathematica Slovaca 50 (2000), 271–287.

[5] M. R. Garey and D.S. Johnson, Crossing number is NP-complete, SIAM J Discrete Math 4 (1983), 312–316.

[6] R. P. Guy, Latest results on crossing numbers, In Proc. Recent Trends in Graph Theory, Lecture Notes in Mathematics 186, 1971, pp. 143–156.

[7] F. Harary, Graph Theory, Addison Wesley, New York, NY, USA, 1969.

[8] F. T. Leighton, Complexity Issues in VLSI, MIT Press, USA, 1983.

[9] A. Liebers, Planarizing graphs—a survey and annotated bibliography, J Graph Algorithms Appl 5 (2001), 1–74.

[10] T. Madej, Bounds for the crossing number of the *n*-cube, J Graph Theory 15 (1991), 81–97.

[11] F. Shahrokhi, O. Sýkora, L. A. Székely, and I. Vrt'o, Crossing numbers: bounds and applications, In: Proc. Intuitive Geometry, I. Bárány and K. Boroczky (Editors), Bolyai Society Mathematical Studies 6, Akadémia Kiadó, Budapest, 1997, pp. 179–206.

[12] O. Sýkora and I. Vrt'o, On the crossing number of the hypercube and cube connected cycles, BIT 33 (1993), 232–237.