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Procedia Computer Science 00 (2025) 000–000

Procedia Computer Science

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XIII Latin American Algorithms, Graphs, and Optimization Symposium (LAGOS 2025)

Type 1 and Type 2 Kochol superposition snarks

Rieli Araújo[a], Celina M.H. de Figueiredo[a], Diana Sasaki[b], Simone Dantas[c]

[a] Rio de Janeiro Federal University, Brazil
[b] Rio de Janeiro State University, Brazil
[c] Fluminense Federal University, Brazil

Abstract

Snarks are a historical class of cubic graphs with peculiar properties motivated by the Four-Color Theorem. In nearly 100 years of search, since its definition by Peter Guthrie Tait in 1880, only five such graphs were identified which motivated Martin Gardner in 1976 to call them snark, a mysterious creature. In 1975, Rufus Isaacs introduced a method known as dot product, which allowed the construction of new snarks from known snarks, and presented the first infinite family of snarks. A new method proposed in 1996 by Martin Kochol allowed to obtaining new snarks from smaller graphs, known as *Kochol superposition*. However, this method was usually used to obtain snarks with large girth. We applied the Kochol superposition to known snarks: the family of Goldberg snarks (known to be Type 1) and a girth 4 snark recently discovered by Gunnar Brinkmann et al. (known to be Type 2). Surprisingly, when we apply the Kochol superposition to a Type 1 snark with a Type 2 snark, we can obtain new families of snarks of distinct Types: Type 1 and Type 2.

Keywords: Total coloring; Kochol superposition; Snarks

1. Introduction

In 1880, Tait [13] proved that the Four-Color Theorem is equivalent to the statement that every bridgeless cubic map is 3-edge-colorable. In this context, emerged the Petersen graph, a bridgeless cubic graph that does not allow a 3-edge-coloring. Later, an intense search for such graphs began, and in about 100 years only five similar graphs were identified. In 1976, Gardner [6], inspired by the difficulty of finding these graphs, named them *snarks*, drawing inspiration from the poem "The Hunting of the Snark" by Lewis Carroll [5]. The Petersen graph was the first discovered snark and it is known to be the smallest one.

* Corresponding author.
E-mail address: rieli@cos.ufrj.br; celina@cos.ufrj ; diana.sasaki@ime.uerj.br; sdantas@id.uff.br

Throughout the studies of these mysterious graphs, some definitions have been developed over time. In this paper, we assume that snarks are cubic, cyclically 4-edge-connected, and Class 2 graphs, i.e., that cannot be 3-edge-colored. A graph is considered cyclically 4-edge-connected when the removal of at least 4 edges is necessary to disconnect it, and the resulting connected components must contain cycles. For further details on snark properties, we refer to [2].

In 1975, Isaacs [8] introduced an operation known as the *dot product*, which allowed the construction of new snarks from known snarks, as well as the creation of the first infinite family of snarks. In 1996, Kochol [9] proposed a new method for obtaining new snarks known as *Kochol superposition*. The Kochol superposition was usually used to construct snarks with large girth. The girth of a graph is the length of its the shortest cycle.

In 2015, Brinkmann, Preissmann and Sasaki [3] studied the total coloring of snarks. A snark is of Type 1 if it has a total coloring of its vertices and edges with four colors; it is of Type 2 if any total coloring requires at least five colors. They constructed the smallest Type 2 snark with girth 4, that we refer to as *brick snark*. We propose applying Kochol superposition to the well-known infinite family of Goldberg snarks [7] and the brick snark, obtaining girth 4 snarks of different types.

2. Preliminaries

A *semi-graph* is a 3-tuple $G = (V, E, S)$, where $V(G)$ is a set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semiedges* having only one endpoint belonging to $V(G)$. When it is clear, we simply denote $V(G)$, $E(G)$ and $S(G)$ by V, E and S, respectively. A *graph* is a semi-graph with a empty set S. A semiedge with endpoint $v \in V$ is denoted by $v\cdot$. An edge $e \in E$ whose endpoints are vertices v and w is denoted by vw. Given two semiedges $v\cdot$ and $w\cdot$, the *junction* of $v\cdot$ and $w\cdot$ is formed by replacing $v\cdot$ and $w\cdot$ with the edge vw. The semi-graphs considered in this paper are undirected and simple,i.e. no loops and no multiple edges.

The *degree* $d(v)$ of a vertex $v \in V$ is the number of elements of $(S \cup E)$ that are incident to v. We say that G is a *cubic graph* or *cubic semi-graph* if the degree of each vertex is equal to three.

Graph coloring involves the partition of its elements (edges, semiedges and/or vertices) into a family of disjoint sets $C = \{c_1, c_2, ..., c_k\}$, called *color classes*, where k is the number of colors assigned to a graph G and c_i with $i = \{1, 2, ...k\}$ is a *color class*.

A *total coloring* is an assignment of colors to its elements of G such that no adjacent edges and semiedges, adjacent vertices, incident vertices or edges and semiedges share the same color. When k colors are assigned to the elements of G, we say that G has a *k-total coloring*. The *total chromatic number* of G is the smallest value of k necessary to obtain a total coloring, denoted by $\chi''(G)$. Clearly, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. Independently, Behzad [1] and Vizing [15] conjectured an upper bound for the total chromatic number.

Conjecture 2.1 (Total Coloring Conjecture (TCC) [15, 1])**.** *For every simple graph G, the total chromatic number of G satisfies $\chi''(G) \leq \Delta(G) + 2$.*

Since $\chi''(G) \geq \Delta(G) + 1$, and according to the Total Coloring Conjecture (TCC), $\chi''(G) \leq \Delta(G) + 2$, it is conventionally established that: when $\chi''(G) = \Delta(G) + 1$, the graph is called *Type 1*; and when $\chi''(G) = \Delta(G) + 2$, the graph is called *Type 2*. Determining whether a graph is Type 1 or Type 2, is an NP-hard problem [12]. Although the conjecture is still open, its validity has been confirmed for some classes of graphs, as occurred for cubic graphs, independently demonstrated by Rosenfeld [11] and Vijayaditya [14]. If G is a cubic graph, then its total chromatic number is either 4 or 5.

2.1. Kochol superposition

We follow the definition of the Kochol superposition construction as presented in [9] and [10]. Given a cubic semi-graph $G(V, E, S)$, with $S \neq \emptyset$, the set S of semiedges is partitioned into n pairwise disjoint non-empty sets $S_1, S_2, \ldots, S_n$, such that $|S_i| = k_i$ for $i = \{1, 2, \ldots, n\}$. Following the notation of [9], the sets S_i are called *connectors*, and the semi-graph G is denoted by $(k_1, k_2, \ldots, k_q)$-semi-graph G.

This method uses two fundamental structures:

A *superedge* ξ is a semi-graph with two connectors, and a *supervertex* ϑ is a semi-graph with three connectors. We consider the following semi-graphs depicted in Figure 1:

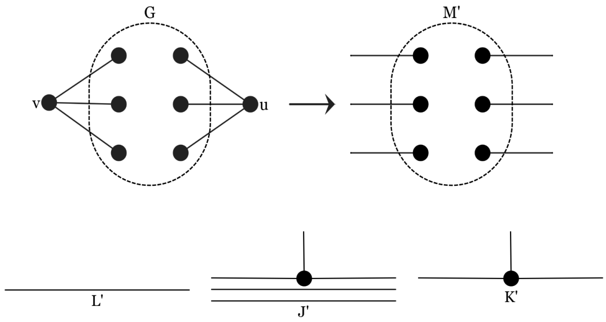


Fig. 1: Superedge M' obtained from snark G, superedge L', supervertex J' and supervertex K'.

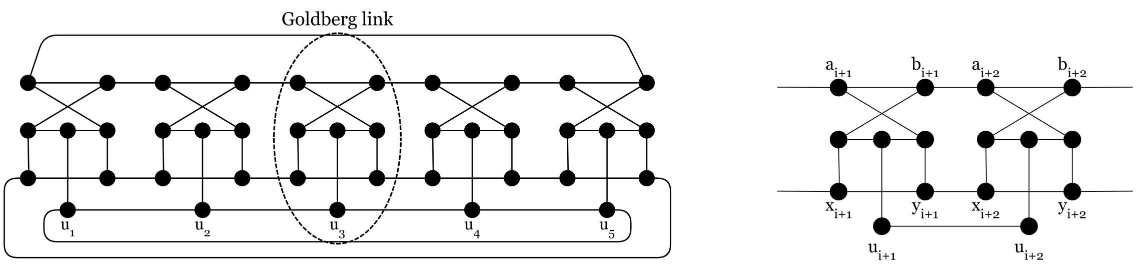
i. $(3,3)$-semi-graph M' (superedge) is obtained by removing two nonadjacent vertices v_1 and v_2 from a snark G, and replacing each edge incident to v_1 or v_2 by semiedges.
ii. $(1,1)$-semi-graph L' (superedge) is an isolated edge (two semiedges);
iii. $(1,3,3)$-semi-graph J' (supervertex) consists of two isolated edges and a vertex;
iv. $(1,1,1)$-semi-graph K' (supervertex) consists of a vertex and three semiedges.

3. Constructing new infinite families of girth 4 snarks by Kochol superposition

In order to apply the Kochol superposition, we consider the known family of Goldberg snarks, which was proved to be Type 1 by Campos et al. [4], along with the girth 4 Type 2 snark of Figure 4 constructed by Brinkmann, Preissmann and Sasaki [3], that we refer to as *brick snark*.

Goldberg snark. The infinite family of Goldberg snarks was created by Goldberg [7]. The first member of this family, snark G_5, has 40 vertices and 60 edges, and it is formed by the junction of semiedges of five semi-graphs referred to as the *Goldberg link*, or simply *link* (Figure 2a). These snarks grow infinitely and recursively by an odd number of connected links, $\ell = 5, 7, 9 \ldots$, with odd $\ell \geq 5$, meaning that G_5 has five links, G_7 has seven links, and so on. Figure 2a shows the G_5 with a Goldberg link highlighted.

The most common way to describe the growth of this family is that each new member is obtained by inserting a *double link* into the previous member. A double link is a semi-graph $\mathcal{L}^* = (V, E, S)$, with 16 vertices, 20 edges and 6 semiedges, obtained from two Goldberg links, as shown in Figure 2b.



(a) First member of Goldberg snark.

(b) Semi-graph double link $\mathcal{L}^*$.

Fig. 2: Snark G_5 (with the Goldberg link highlighted with a dashed ellipse), and semi-graph $\mathcal{L}^*$.

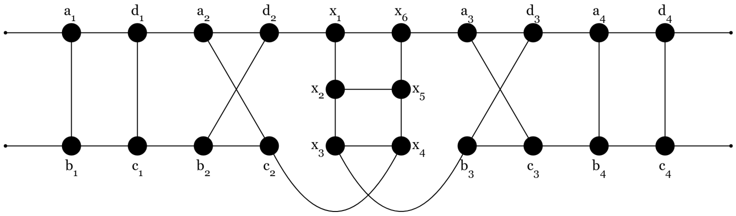


Fig. 3: Brick semi-graph B^*.

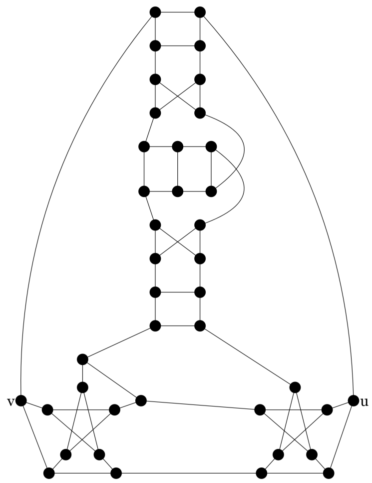


Fig. 4: Brick snark: Type 2 snark with girth 4.

Brick snark. A brick is a cubic semi-graph $B^* = (V, E, S)$ with four pairwise non-adjacent semiedges and whose underlying graph $H = (V, E)$ is a subgraph of some cyclically 4-edge-connected cubic graph. For any semi-graph $G = (V, E, S)$, we call the graph H as the *underlying graph* of G [3].

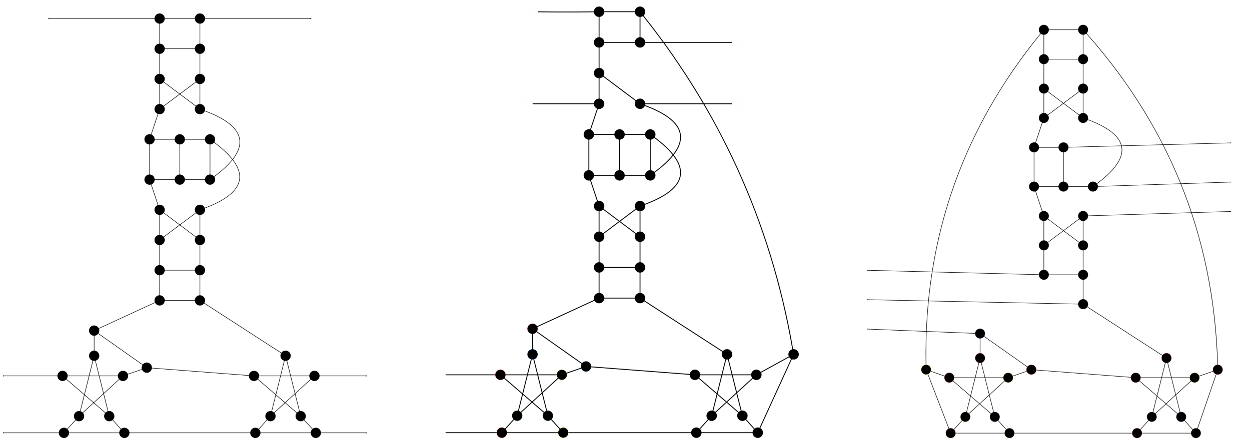
Brinkmann et al. [3] proved that the brick semi-graph B^*, shown in Figure 3, with 22 vertices and girth 4 is the smallest Type 2 brick. They also established that the snark shown in Figure 4, defined as *Brick snark*, with 40 vertices (including $V(B^*)$) and girth 4, is the smallest Type 2 snark.

3.1. Our new infinite families by Kochol superposition

We propose using the brick snark to construct superedges (ξ) for the Kochol superposition. For this purpose, it is necessary to remove an arbitrary pair of two non-adjacent vertices. In the brick snark, there are $C_2^{40} = 780$ possible ways to choose an arbitrary pair of two distinct vertices. However, this choice is restricted to non-adjacent vertices. Since the brick snark has 60 edges, we estimate the number of valid choices to be 720. Nevertheless, some superedges may be isomorphic to others, so we allow the generation of at most 720 different superedges. In Figure 5, we show three different superedges created by the brick snark, each of which generates a new family obtained through the Kochol superposition.

The Kochol superposition construction is presented as follows: The vertices $\{u_1, u_2, \ldots, u_n\}$ of the cycle in the Goldberg snarks (see in Figure 2) constitute the supervertices ϑ, and each edge of this cycle is replaced by a superedge ξ from the brick snark, with junctions made between the semiedges of the superedges and the supervertices accordingly, resulting in a new family denoted by $\mathcal{GB}$. We show the first member $\mathcal{GB}_1$ and double link $\mathcal{L}_{\mathcal{GB}}$, in Figure 6.

This new family follows the same growth pattern as the Goldberg family, i.e. each successive member is constructed by inserting of $\mathcal{L}_{\mathcal{GB}}$ (shown in Figure 6b) into the previous member. Observe that each superedge used in the Kochol superposition construction gives rise to a new family, resulting in a finite set of distinct infinite families.



(a) ξ_1 (b) ξ_2 (c) ξ_3

Fig. 5: Three distinct superedges ξ_1, ξ_2 and ξ_3 created by the removal of no vertex, one vertex, or two vertices from $V(B^*)$ in the brick snark, respectively.

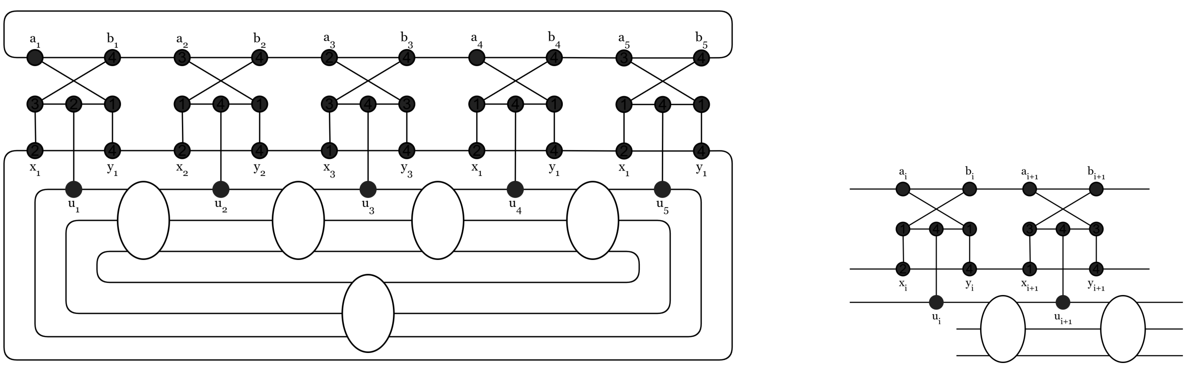
In Figure 5, we present three different superedges, and in every case, girth 4 has been preserved. Clearly, removing an arbitrary pair of two vertices from the brick snark to generate a superedge ensures that the resulting superedge has girth 4, which leads to Theorem 3.1.

Theorem 3.1. *A Kochol superposition of the Goldberg snarks with the brick snark generates new infinite families of snarks with girth 4.*

Proof. Since the Goldberg snarks have girth 5, the Kochol superposition construction does not create graphs with smaller girth. Moreover, the brick snark has girth 4, and this girth is preserved in the superedges. Therefore, the result follows. We observe that the graph contains six 4-cycles, and therefore the removal of only two vertices would not increase the girth of the graph, thus preserving a girth of size 4. □

4. The total colorings of the constructed snarks

Determining whether all graphs of $\mathcal{GB}$ are Type 1 or Type 2 depends on the superedge ξ. In this way, three scenarios may occur:



(a) First member $\mathcal{GB}_1$ of family $\mathcal{GB}$. (b) Double link $\mathcal{L}_{\mathcal{GB}}$.

Fig. 6: The construction of family $\mathcal{GB}$.

- No vertex from $V(B^*)$ in the brick snark is removed to create the superedge (see Figure 5a);
- Only one vertex from $V(B^*)$ in the brick snark is removed to create the superedge (see Figure 5b);
- Two vertices from $V(B^*)$ in the brick snark are removed to create the superedge (see Figure 5c).

When no vertex is removed from $V(B^*)$ in the brick snark, the resulting superedge ξ is Type 2. This ensures that the Kochol superposition of superedge ξ preserves the Type 2 property. However, if at least one vertex of $V(B^*)$ in the brick snark is removed, the resulting superedge ξ is Type 1. Note that, removing any vertex or edge from $V(B^*)$ in the brick snark yields a Type 1 semi-graph. However, this condition alone does not ensure that the graph constructed using these superedges is Type 1, since it is necessary to find 4-total colorings for the superedges such that, when joined with the Goldberg snark, the resulting graph admits a 4-total coloring. Therefore, some of the snarks obtained are Type 1 while others are Type 2, resulting in Theorem 4.1.

Theorem 4.1. *A Kochol superposition of the Goldberg snarks with the brick snark generates new infinte families of Type 1 and Type 2 snarks.*

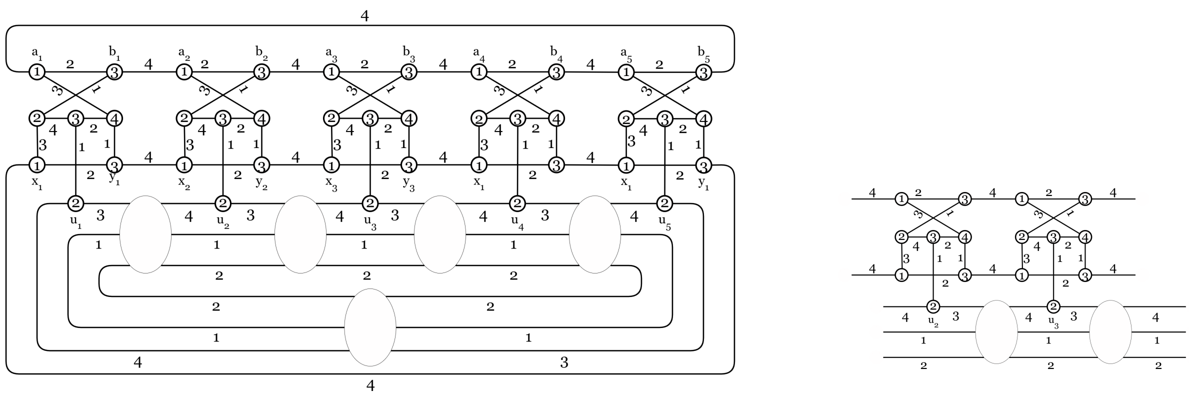
Proof. First, we construct a partial 4-total coloring for all members of the families $\mathcal{GB}$, as presented in Figure 7. Remark that each superedge ξ_1, ξ_2, and ξ_3 gives a different infinite family $\mathcal{GB}$.

Next, we analyze the total colorings of graphs $\mathcal{GB}_1$, where the choice of vertices removed to create the superedge is crucial in determining whether the snarks generated by Kochol superposition are Type 1 or Type 2. The analysis is divided into two cases:

Case 1 (If no vertex from $V(B^*)$ in the brick snark is selected to create the superedges):
In this case, since the Type 2 subgraph is preserved, any graph generated is Type 2. In Figure 8a, we show a 5-total coloring of superedge ξ_1 obtained by removing two vertices that do not belong to B^*, and that is compatible with the 4-total coloring of the Goldberg snarks. We say that two colorings are compatible if they assign the same color to corresponding semiedges and different colors to their incident vertices. Thus, the Kochol superposition applied to the Goldberg snark and the superedge ξ_1 generates a new infinite family, denoted by $\mathcal{GB}^1$, in which all members are Type 2 graphs.

Case 2 (If at least one vertex from $V(B^*)$ in the brick snark is selected to create the superedges):

In this case, it is possible to obtain a 4-total coloring for the resulting superedge. Remark that the determined 4-total colorings must be compatible with the 4-total coloring of the Goldberg snarks, ensuring that all members of the $\mathcal{GB}$ family admit 4-total colorings. In Figures 8b and 8c, we present the superedges ξ_2 and ξ_3 with the used 4-total



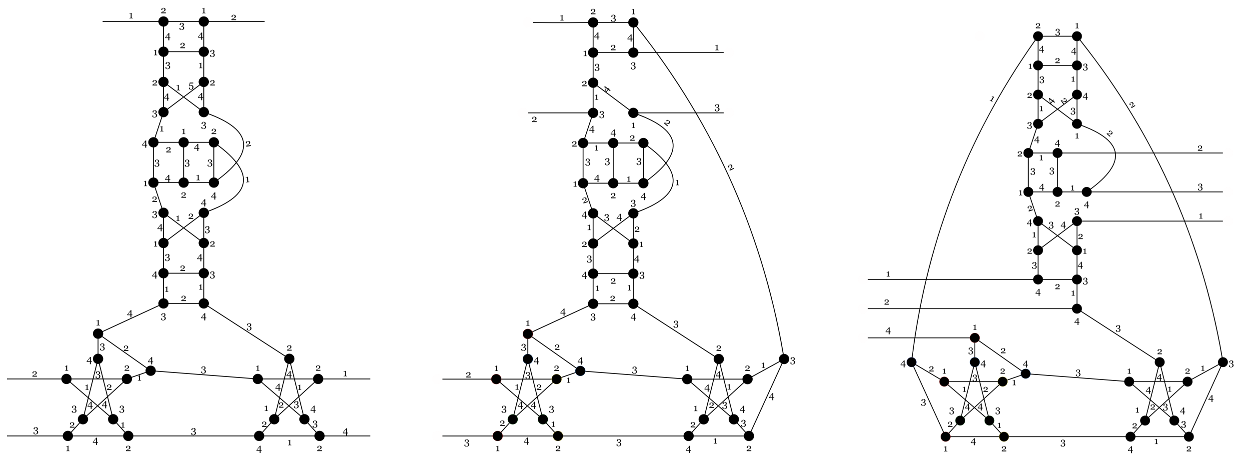
(a) Graph $\mathcal{GB}_1$ with a partial 4-total coloring (the superedges are not colored).

(b) Double link $\mathcal{L}_{\mathcal{GB}}$ with a partial 4-total coloring (the superedges are not colored).

Fig. 7: The construction of total colorings of graphs $\mathcal{GB}$.

colorings. Each of these superedges generates a new infinite family in which all members are Type 1, denoted by $\mathcal{GB}^2$ and $\mathcal{GB}^3$, respectively.

Therefore, we have shown that it is possible to generate Type 1 and Type 2 snark graphs by applying the Kochol superposition to the Goldberg snarks and the brick snark, thus preserving the snark properties with girth 4. □



(a) A 5-total coloring of superedge ξ_1. (b) A 4-total coloring of superedge ξ_2. (c) A 4-total coloring of superedge ξ_3.

Fig. 8: The superedges with the total colorings.

5. Conclusion

This work constructs new infinite families of snarks and establishes that the snarks obtained through this construction are Type 1 or Type 2, depending on the choice of vertices removed during the application of the Kochol superposition. This result expands the understanding of total coloring properties in snarks and provides a method for generating new snarks of different Types from a single construction. Future research could investigate how this method can be extended to other families of snarks and how the Kochol superposition affects the total coloring when all parts of the Kochol superposition are Type 1 separately, but when joined with this method, we might construct a Type 2 graph.

Acknowledgements

Partially supported by CAPES – Finance Code 001; CNPq 311260/2021-7; FAPERJ SEI-260003/014835/2023, E-26/210.649/2023.

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