Systems Engineering and Computer Science Graduate Program

Alberto Luiz Coimbra Institute for Graduate Studies and Research in Engineering

FEDERAL UNIVERSITY OF RIO DE JANEIRO

TECHNICAL REPORT

TR-PESC/Coppe/UFRJ- 793/2024

Numerically Checking the Dislocation Hyperbolic Augmented Lagrangian Algorithm for Nonconvex Optimization Problems

Authors:

LENNIN MALLMA RAMIREZ[1*], IGOR P.S. PEREIRA [2†], NELSON MACULAN[1†], ARGIMIRO R. SECCHI[2†]

[1*]PROGRAMA DE ENGENHARIA DE SISTEMAS E COMPUTAÇÃO, COPPE, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, RIO DE JANEIRO, BRAZIL.

[2]PROGRAMA DE ENGENHARIA QUÍMICA, COPPE, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, RIO DE JANEIRO, BRAZIL.

[*]CORRESPONDING AUTHOR(S). E-MAIL(S): LENNINMR@COS.UFRJ.BR ; CONTRIBUTING AUTHORS: IPEREIRA@PEQ.COPPE.UFRJ.BR; MACULAN@COS.UFRJ.BR; ARGE@PEQ.COPPE.UFRJ.BR;

[†]THESE AUTHORS CONTRIBUTED EQUALLY TO THIS WORK.

Abstract

In this note, we ensure that the dislocation hyperbolic augmented Lagrangian algorithm converges to a global minimizer, assuming nonconvexity assumptions. The subproblem generated by this algorithm is solved with the DIRECT algorithm. Finally, we present computational experiments to show the proposed algorithm's good performance.

Keywords: Augmented Lagrangian, Box-constrained, Nonconvex problem, Deterministic algorithm

Publication Date: 18/09/2024

Numerically Checking the Dislocation Hyperbolic Augmented Lagrangian Algorithm for Nonconvex Optimization Problems

Lennin Mallma Ramirez[1*], Igor P.S. Pereira [2†], Nelson Maculan[3†], Argimiro R. Secchi[4†]

[1*]Programa de Engenharia de Sistemas e Computação , COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
[2,4] Programa de Engenharia Química, COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.
[3]Programa de Engenharia de Sistemas e Computação , COPPE, Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil.

*Corresponding author(s). E-mail(s): lenninmr@cos.ufrj.br ;
Contributing authors: ipereira@peq.coppe.ufrj.br; maculan@cos.ufrj.br; arge@peq.coppe.ufrj.br;
†These authors contributed equally to this work.

Abstract

In this note, we ensure that the dislocation hyperbolic augmented Lagrangian algorithm converges to a global minimizer, we assuming nonconvexity assumptions. The subproblem generated by this algorithm is solved with the DIRECT algorithm. Finally, we present computational experiments to show the good performance of the proposed algorithm.

Keywords: Augmented Lagrangian, Box-constrained, Nonconvex problem, Deterministic algorithm

1 Introduction

In this note, we are interested in solving the following inequality constrained nonconvex optimization problem:

$$(P) \qquad min\ f(x) \quad s.t. \quad g(x) \leq 0, \quad x \in \Omega,$$

where the functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuously differentiable, and let $a_1, a_2 \in \mathbb{R}^n$ such that $\Omega = \{x \in \mathbb{R}^n \ : \ -\infty < a_1 \leq x \leq a_2 < \infty\}$, is a nonempty closed subset in $\mathbb{R}^n$. In particular, augmented Lagrangian type algorithms solve the problem (P), examples are: Lagrangian function based on the quadratic penalty [2] and class of nonquadratic Lagrangian functions [7]. In [7], an interesting feature of these algorithms is the detection of infeasibility. The study of this class of Lagrangian algorithms can be seen in paper [10].

The hyperbolic penalty function was used in the hyperbolic penalty algorithm proposed by [14]. The dislocation hyperbolic augmented Lagrangian function (DHALF) was proposed in [15]. This type of hyperbolic function caught the attention of different researchers, thus, works appeared from the computational point of view, see [1], [5] and [13]; and theoretical works, see [6] and [4]. All this motivates us to continue investigating this function and propose our dislocation hyperbolic augmented Lagrangian algorithm to find global solutions to nonconvex optimization problems. Some characteristics of our algorithm are that we consider a nonquadratic penalty, and the safeguard technique to update the Lagrange multipliers is not considered. The contributions of our work are as follows:

- This work contributes to ensuring that the sequence generated by our proposed algorithm converges to a global solution of the problem (P) using DHALF. To solve the box-constrained subproblem generated by our algorithm, we use the global deterministic optimization algorithm called DIRECT (see [8] and [9]).

The note is organized as follows: In Section 2, we state some definitions and basic results that are used during this work and we present our proposed algorithm. In Section 3, a convergence result is proposed. In Section 4, some computational examples are presented in order to show how our proposed algorithm works.

2 Deterministic Global Dislocation Hyperbolic Augmented Lagrangian Algorithm (DGDHALA)

The Lagrangian function of problem (P) is defined by $L : \mathbb{R}^n \times \mathbb{R}^m_+ \to \mathbb{R}$,

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x),$$

for $x \in \Omega$ and where $\lambda_i \geq 0, \ i = 1, ..., m$, they are the Lagrange multiplier. We define the dislocation hyperbolic penalty function (DHPF) related to the constraints of problem (P), as follows, $p : \mathbb{R} \times \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$

$$p(g_i(x), \lambda_i, \tau) = \tau \left(\frac{\lambda_i g_i(x)}{\tau} + \sqrt{\left(\frac{\lambda_i g_i(x)}{\tau}\right)^2 + 1} - 1 \right)$$

$$= \tau h \left(\frac{\lambda_i g_i(x)}{\tau} \right), \; i = 1, ..., m, \tag{2.1}$$

where the function $h : \mathbb{R} \to \mathbb{R}$, is defined as $h(t) = t + \sqrt{t^2 + 1} - 1$. This function is a smoothing of the exact penalty function studied by [16]. Henceforth, we will call the h function as dislocation hyperbolic function (DHF). In the following, we are going to present some properties of h: (H1) $h(0) = 0$ and $h'(0) = 1$; (H2) $h'(t) = 1 + \frac{t}{\sqrt{t^2+1}} > 0, \;\; \forall t \in \mathbb{R}$; (H3) $h''(t) = \frac{1}{(t^2+1)^{\frac{3}{2}}} > 0, \;\; \forall t \in \mathbb{R}$.

We are going to consider function (2.1) to define the DHALF of problem (P) by $l_H : \mathbb{R}^n \times \mathbb{R}^m_{++} \times \mathbb{R}_{++} \to \mathbb{R}$,

$$l_H(x, \lambda, \tau) = f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i, \tau) = f(x) + \sum_{i=1}^{m} \tau \, h \left(\frac{\lambda_i g_i(x)}{\tau} \right). \tag{2.2}$$

Next, we introduce our algorithm DGDHALA.

2.1 Algorithm DGDHALA

Step 1. Let $x^0 \in \mathbb{R}^n$, $\lambda^0 = (\lambda^0_1, ..., \lambda^0_m) \in \mathbb{R}^m_{++}$, $\tau^0 \in \mathbb{R}_{++}$, $1 < \alpha$, and $0 < \theta < 1$. Compute $W^0_i = \max\{g_i(x^0), 0\}$. Set $k = 1$.

Step 2. Find $x^k \in \mathbb{R}^n$ as an approximate global solution of the problem

$$min_{x \in \Omega} \; l_H(x, \lambda^k, \tau^k). \tag{2.3}$$

Step 3. Updating of Lagrange multiplier. Compute

$$\lambda^{k+1}_i = \lambda^k_i h' \left(\frac{\lambda^k_i g_i(x^k)}{\tau^k} \right), \;\; i = 1, ..., m. \tag{2.4}$$

Step 4. Update penalty parameter. Compute

$$W^k_i = min\{-g_i(x^k), \; \lambda^k_i\}, i = 1, ..., m.$$

If,

$$\left\| W^k \right\|_\infty \leq \theta \left\| W^{k-1} \right\|_\infty, \tag{2.5}$$

set, $\tau^{k+1} = \tau^k$. Otherwise, set $\tau^{k+1} = \alpha \tau^k$.

Step 5. $k := k + 1$. Go to Step 2.

In **Step 2**, we are going to consider the DIRECT algorithm to solve the subproblem (2.3), which is a global deterministic optimization algorithm for box-constrained problems. In **Step 3**, the new estimates of the multipliers are updated. In **Step 4**, the condition (2.5) considered to update the penalty parameter, this condition is also studied by [5]. Throughout our work, we will consider the following assumption

$$\lim_{k\to+\infty} \tau^k = \tilde{\tau} > 0,$$

that is, the sequence $\{\tau^k\}$ converges to a strictly positive point.

2.2 Characteristics of DGDHALA

For $x \in \mathbb{R}^n$, we define the following sets of indices
$I_0 = \{i \in \{1, ..., m\} \mid g_i(x) = 0\}$ and $I_- = \{i \in \{1, ..., m\} \mid g_i(x) < 0\}$. Let us consider the following cases: (c1) If $i \in I_0$, then we have at the k-th iteration $g_i(x^k) = 0$, then by (2.4) and (H1), we get, $\lambda_i^{k+1} = \lambda_i^k$; (c2) If $i \in I_-$, then we have at the k-th iteration $g_i(x^k) < 0$, then by (2.4), (H3), (H1), and by a development similar to the previous case we can obtain $\lambda_i^k > \lambda_i^{k+1}$. The following result ensures the feasibility of the estimated multipliers.

Proposition 1. *Let* $\left\{\lambda^k = (\lambda_1^k, ..., \lambda_m^k) \mid k = 1, 2, ...\right\} \subset \mathbb{R}^m$. *If*

$$\lambda^k \in \mathbb{R}^m_{++} \quad \textit{then} \quad \lambda^{k+1} \in \mathbb{R}^m_{++}.$$

Proof. See Remark 3.3 of [12]. □

3 Convergence

In this section, we present our main results. Let us consider the following assumption:

(A1) For all $k \in I\!N$, we obtain $x^k \in \Omega$ such that

$$l_H(x^k, \lambda^k, \tau^k) \le l_H(x, \lambda^k, \tau^k) + \epsilon^k, \ \ \forall\, x \in \Omega,$$

where the sequence of tolerances $\{\epsilon^k\} \subset \mathbb{R}_+$ is bounded.

The assumption **(A1)** is considered in Chapter 5 of [3], to ensure global minimization. The following results are similar to Theorem 5.1 and Theorem 5.2 of [3] (see also [2] and [5]). In the following result, we ensure that the sequence generated by the algorithm DGDHALA converges to a feasible point.

Theorem 2. *The sequences* $\{x^k\}$ *and* $\{\lambda^k\}$ *are generated by DGDHALA. Suppose that the sequence* $\{\lambda^k\}$ *is bounded and the whole sequence* $\{x^k\}$ *is convergent, i.e.,* $\lim_{k\to\infty} x^k = x^*$. *Then, every limit point* x^* *is a feasible point.*

Proof. Let $x^k \in \Omega$ and we know that the set Ω is closed, then $x^* \in \Omega$. Since that $\{\tau^k\}$ is bounded, then there exists $k_0 \in I\!N$ such that $\tau^k = \tau^{k_0} = \tilde{\tau} > 0$ for all, $k \geq k_0$. Then the condition (2.5) is verified for all $k \geq k_0$. Thus,

$$\left\|W^k\right\|_\infty \to 0, \tag{3.6}$$

so,

$$-g_i(x^k) \to 0, \quad or \quad \lambda_i^k \to 0, \;\; with, \;\; g_i(x^k) < 0, \; i = 1, ..., m, \tag{3.7}$$

this implies

$$g_i(x^k)_+ \to 0, \; i = 1, ..., m,$$

and we obtain that the limit point is feasible. □

Now, we ensure that the sequence generated by the algorithm DGDHALA converges to a global solution.

Theorem 3. *The sequence $\{x^k\}$ and $\{\lambda^k\}$ are generated by DGDHALA. Assume that the sequence $\{\lambda^k\}$ is bounded, $\lim_{k\to\infty} \epsilon^k = 0$, the problem (P) has a non-empty set of feasible solutions and the whole sequence $\{x^k\}$ is convergent, i.e., $\lim_{k\to\infty} x^k = x^*$. Then, x^* is a global minimizer of the problem (P).*

Proof. The whole sequence is convergent, i.e., $\lim_{k\to\infty} x^k = x^*$, where x^* is feasible by Theorem 2 since the problem (P) has a non-empty set of feasible solutions. Since $\{\tau^k\}$ is bounded, then there exists $k_0 \in I\!N$ such that $\lim_{k\geq k_0} \tau^k = \tau^{k_0} = \tilde{\tau} > 0$, for all, $k \in K \subset I\!N$, $k \geq k_0$ (with $x \in \Omega$), By **(A1)** we have,

$$f(x^k) + \sum_{i=1}^{m} \tau^{k_0} h\left(\frac{\lambda_i^k g_i(x^k)}{\tau^{k_0}}\right)$$

$$\leq f(x) + \sum_{i=1}^{m} \tau^{k_0} h\left(\frac{\lambda_i^k g_i(x)}{\tau^{k_0}}\right) + \epsilon^k, \; \forall k \geq k_0. \tag{3.8}$$

Since we have $g(x) \leq 0$, we can get,

$$\frac{\lambda_i^k g_i(x)}{\tau^{k_0}} \leq 0, \; i = 1, ..., m, \; \forall k \geq k_0,$$

we apply (H2) in the inequality above, so

$$\tau^{k_0} \; h\left(\frac{\lambda_i^k g_i(x)}{\tau^{k_0}}\right) \leq \tau^{k_0} \; h(0), \; i = 1, ..., m, \; \forall k \geq k_0,$$

by (H1), it follows $\tau^{k_0} \; h\left(\frac{\lambda_i^k g_i(x)}{\tau^{k_0}}\right) \leq 0, \; i = 1, ..., m, \; \forall k \geq k_0$, we rewrite as follows

$$\sum_{i=1}^{m} \tau^{k_0} \; h\left(\frac{\lambda_i^k g_i(x)}{\tau^{k_0}}\right) \leq 0, \; \forall k \geq k_0. \tag{3.9}$$

By (3.9) in (3.8), we have

$$f(x^k) + \sum_{i=1}^{m} \tau^{k_0} h\left(\frac{\lambda_i^k g_i(x^k)}{\tau^{k_0}}\right) \leq f(x) + \epsilon^k, \ \forall k \geq k_0. \tag{3.10}$$

Since $\{\lambda^k\}$ is bounded, then there exists a infinite subset of indices $K_1 \subset K$, and by Proposition 1 we have

$$\lim_{k \in K_1} \lambda_i^k = \lambda_i^* \geq 0, \ i = 1, ..., m.$$

We know that $g_i(x^*) \leq 0, i = 1, ..., m$, and taking limits in the inequality above (3.10) for $k \in K_1$, we have

$$f(x^*) + \sum_{i=1}^{m} \tau^{k_0} h\left(\frac{\lambda_i^* g_i(x^*)}{\tau^{k_0}}\right) \leq f(x).$$

We rewrite the above as

$$f(x^*) + \sum_{i \in I_-(x^*)} \tau^{k_0} h\left(\frac{\lambda_i^* g_i(x^*)}{\tau^{k_0}}\right) + \sum_{i \in I_0(x^*)} \tau^{k_0} h\left(\frac{\lambda_i^* g_i(x^*)}{\tau^{k_0}}\right) \leq f(x),$$

considering (H1) in the case $i \in I_0(x^*)$, in the inequality above, therefore we only have

$$f(x^*) + \sum_{i \in I_-(x^*)} \tau^{k_0} h\left(\frac{\lambda_i^* g_i(x^*)}{\tau^{k_0}}\right) \leq f(x), \ \forall k \geq k_0. \tag{3.11}$$

On the other hand, since that $\{\tau^k\}$ is bounded and since we have $g_i(x^*) < 0$ in (3.11) and by (2.5) we obtain

$$\lim_{k \in K_1} \lambda_i^k = \lambda_i^* = 0. \tag{3.12}$$

Then, from (3.11) and (3.12) we obtain

$$f(x^*) + \sum_{\{i: g_i(x^*) < 0\}} \tau^{k_0} h(0) \leq f(x),$$

from (H1) and the inequality above it follows that $f(x^*) \leq f(x)$, where x is an arbitrary feasible point. $\square$

4 Computational Examples

Let us consider the following examples to verify the performance of our proposed algorithm.

Example 1. *See Problem 2 of [18], Example 3 of [17].*

$$\min_{x\in\mathbb{R}^2} f(x) = -x_1 - x_2$$
$$s.t. \quad g_1(x) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_2 - 2 \leq 0,$$
$$g_2(x) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leq 0,$$
$$x \in \Omega = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 3,\ 0 \leq x_2 \leq 4\}.$$

The global solution is $\tilde{x} = (2.3295, 3.1783)^T$ *with optimal value* $f(\tilde{x}) = -5.5079$.

Example 2. *See Example 2 of [17].*

$$\min_{x\in\mathbb{R}^3} f(x) = -x_1x_2x_3$$
$$s.t. \quad g_1(x) = x_1 + 2x_2 + 2x_3 - 72 \leq 0,$$
$$g_2(x) = -x_1 - 2x_2 - 2x_3 \leq 0,$$
$$x \in \Omega = \{x \in \mathbb{R}^3 : 0 \leq x_i \leq 42,\ i = 1, 2, 3\}.$$

The global solution is $\tilde{x} = (24, 12, 12)$ *with function value* $f(\tilde{x}) = -3456$.

Example 3. *See Example 6.1 of [11]*

$$\min_{x\in\mathbb{R}^5} f(x) = c^Tx - 0.5x^TQx$$
$$s.t. \quad g_1(x) = 20x_1 + 12x_2 + 11x_3 + 7x_4 + 4x_5 - 40 \leq 0,$$
$$x \in \Omega = \{x \in \mathbb{R}^5 : 0 \leq x_i \leq 1,\ i = 1, ..., 5\},$$

where $c = (42, 44, 45, 47, 47.5)^T$, $Q = 100I$, *and* I *is the identity matrix. The global solution is* $\tilde{x} = (1, 1, 0, 1, 0)^T$ *with function value* $f(\tilde{x}) = -17$.

Example 4. *See Problem 2 of [7].*

$$\min_{x\in\mathbb{R}^2} f(x) = -x_1 - x_2$$
$$s.t. \quad g_1(x) = x_1x_2 - 4 \leq 0,$$
$$x \in \Omega = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 6,\ 0 \leq x_2 \leq 4\}.$$

The problem possesses two strict local minima at points $\tilde{x} = (1, 4)$ *and* $\tilde{x} = (6, 0.666667)$ *with objective function values of* -5 *and* -6.66667, *respectively.*

Example 5. *See Problem 5 of [7]*

$$\min_{x\in\mathbb{R}^2} f(x) = x_1^4 - 14x_1^2 + 24x_1 - x_2^2$$
$$s.t. \quad g_1(x) = -x_1 + x_2 - 8 \leq 0,$$
$$g_2(x) = x_2 - x_1^2 - 2x_1 + 2 \leq 0,$$
$$x \in \Omega = \{x \in \mathbb{R}^2 : -8 \leq x_1 \leq 10,\ 0 \leq x_2 \leq 10\}.$$

The global solution is $\tilde{x} = (-3.173599, 1.724533)$ *with* $f(\tilde{x}) = -118.704860$.

4.1 Results

The algorithms were coded in Python and run on a Compaq, Microsoft Windows 10 home, with Intel(R) Core(TM) i3-5005U CPU@2.00GHz, 2000 Mhz. We consider as stopping criteria, the condition of feasibility and complementarity, as follows: $\left\|\lambda^{k+1}g(x^k)\right\|_{l_1} \leq \epsilon_{com}$ and $\left\|g(x^k)\right\|_{l_1} \leq \epsilon_{cons}$, where ϵ_{com} and ϵ_{cons} are tolerances.

- Example 1:
  Initial values: $x^0 = (0,0)$, $\lambda^0 = (1,1)$, $\epsilon_{cons} = 10^{-7}$ and $\epsilon_{com} = 10^{-5}$, $\tau^0 = 0.000002$, $\theta = 0.5$ and $\alpha = 2.5$.

  The results obtained were the following:
  Number of iterations used: 1,
  $x^* = (2.32952576,\ 3.17846635)$,
  $\lambda^* = (3.83943892 \times 10^{-4},\ 7.21484914 \times 10^{-1})$,
  $\tau = 4.9999999999999996 \times 10^{-6}$,
  $f(x^*) = -5.50799210699463$,
  $\|\tilde{x} - x^*\|_2 = 0.0001683327065662843$,
  $f(\tilde{x}) - f(x^*) = 9.21069946295816 \times 10^{-5}$,
  $time\ (s) : 0.09375$.

- Example 2:
  Initial values: $x^0 = (0,0,0)$, $\lambda^0 = (1,1)$, $\epsilon_{cons} = 10^{-5}$ and $\epsilon_{com} = 10^{-3}$, $\tau^0 = 0.000002$, $\theta = 0.25$ and $\alpha = 2$.

  The results obtained were the following:
  Number of iterations used: 9,
  $x^* = (24.06582669, 11.92485902,\ 12.04221917)$,
  $\lambda^* = (1.76677387,\ 1.11022301 \times 10^{-16})$,
  $\tau = 0.000512$,
  $f(x^*) = -3455.89520954783$,
  $\|\tilde{x} - x^*\|_2 = 0.10845173261504679$,
  $f(\tilde{x}) - f(x^*) = 0.10479045217016392$,
  $time\ (s) : 1.328125$.

- Example 3:
  Initial values: $x^0 = (0,0,0,0,0)$, $\lambda^0 = 1$, $\epsilon_{cons} = 10^{-7}$ and $\epsilon_{com} = 10^{-5}$, $\tau^0 = 0.00002$, $\theta = 0.5$ and $\alpha = 2.5$.

  The results obtained were the following:
  Number of iterations used: 1.
  $x^* = (9.99771376 \times 10^{-1},\ 9.99771376 \times 10^{-1},\ 6.85871056 \times 10^{-4}, 9.99771376 \times 10^{-1},\ 2.28623685 \times 10^{-4})$,
  $\lambda^* = 1.99817163 \times^{-10}$,
  $\tau = 5 \times 10^{-5}$,
  $f(x^*) = -16.920129996661075$,

$\|\tilde{x} - x^*\|_2 = 0.0008243146819810678$,
$f(\tilde{x}) - f(x^*) = 0.07987000333892524$,
$time\ (s) : 0.078125$.

- Example 4:
  Initial values: $x^0 = (0,0)$, $\lambda^0 = 1$, $\epsilon_{cons} = 10^{-7}$, $\epsilon_{com} = 10^{-5}$, $\tau^0 = 0.0000002$, $\theta = 0.5$ and $\alpha = 2.5$.

  The results obtained were the following:
  Number of iterations used: 1,
  $x^* = (5.99999812,\ 0.66666667)$,
  $\lambda^* = 0.01247206$,
  $\tau = 5 \times 10^{-7}$,
  $f(x^*) = -6.666664784990243$,
  $\|\tilde{x} - x^*\|_2 = 1.908743042017182 \times 10^{-6}$,
  $f(\tilde{x}) - f(x^*) = 5.215009756476263 \times 10^{-6}$,
  $time(s) : 0.03125$.

- Example 5:
  Initial values: $x^0 = (0,0)$, $\lambda^0 = (1,1)$, $\epsilon_{cons} = 10^{-7}$, $\epsilon_{com} = 10^{-5}$, $\tau^0 = 0.00001$, $\theta = 0.5$ and $\alpha = 2.5$.

  The results obtained were the following:
  Number of iterations used: 3,
  $x^* = (-3.17283951,\ 1.72122901)$,
  $\lambda^* = (5.06159559 \times 10^{-11},\ 3.36584419)$,
  $\tau = 6.25 \times 10^{-5}$,
  $f(x^*) = -118.70483724449679$,
  $\|\tilde{x} - x^*\|_2 = 0.003390158547944282$,
  $f(\tilde{x}) - f(x^*) = 2.275550320973707 \times 10^{-5}$,
  $time(s) : 0.125$.

In work [7], 5 augmented Lagrangian algorithms were studied; in [2] only one algorithm was studied; in [11] they study 4 algorithms and in [17] they study 4 algorithms. In Table 1, we are only going to place the best results obtained in [7], [2], [11] and [17]. In Table 1, the symbol “−” means that the respective example was not solved in the respective work.

Acknowledgments This work has been partially supported by FAPERJ (grant E-26/205.684/2022), COPPETEC Foundation, and CNPq (grants 302435/2019-0).

References

[1] Bagirov A.M., Ordin B., Ozturk G., Xavier A.E.: An incremental clustering algorithm based on hyperbolic smoothing. Computational Optimization and

Table 1 Number of iterations

k	DGDHALA	[7]	[2]	[11]	[17]
Example 1	1	–	–	4	3
Example 2	9	–	–	–	4
Example 3	1	–	–	2	–
Example 4	1	3	5	–	–
Example 5	3	4	3	–	–

Applications. **61**, 219-241 (2015)

[2] Birgin E.G., Floudas C.A., Martínez J.M.: Global minimization using an augmented Lagrangian method with variable lower-level constraints. Mathematical Programming. **125**, 139–162 (2010)

[3] Birgin E.G., Martínez J.M.: Practical Augmented Lagrangian Methods for Constrained Optimization. Vol. 10. SIAM, Philadelphia. (2014)

[4] Birgin E.G., Haeser G., Maculan N., Mallma Ramirez L.: On the global convergence of a general class of augmented Lagrangian methods. Submitted, (2024)

[5] Costa M.F.P., Rocha M.A.M., Fernandes E.M.: An artificial fish swarm algorithm based hyperbolic augmented Lagrangian method. Journal of Computational and Applied Mathematics, **259**, 868–876 (2014)

[6] Gonzaga C.C., Castillo R.A.: A nonlinear programming algorithm based on non-coercive penalty functions. Math. Program., **96**, 87–101 (2003)

[7] Gonçalves M.L.N., Melo J.G., Prudente L.F.: Augmented Lagrangian methods for nonlinear programming with possible infeasibility. J Glob Optim, **63**, 297–318 (2015)

[8] Jones D.R., Perttunen C.D., Stuckman B.E.: Lipschitzian optimization without the Lipschitz constant. Journal of Optimization Theory and Applications. **79**, 157-181, (1993)

[9] Jones D.R., Martins J.R.: The direct algorithm: 25 years later. Journal of Global Optimization. **79**, 521-566, (2021)

[10] Kort B.W., Bertsekas D.P.: Combined primal-dual and penalty methods for convex programming. SIAM J. Control and Optimization. **14**, 268–294 (1976)

[11] Luo H., Sun X., Wu H.: Convergence properties of augmented Lagrangian methods for constrained global optimization. Optimization Methods and Software. **23**, 763-778 (2008)

[12] Mallma Ramirez L., Maculan N., Xavier A.E., Xavier V.L.: Dislocation hyperbolic augmented Lagrangian algorithm for nonconvex optimization. RAIRO-Oper. Res., **57**, 2941–2950 (2023)

[13] Rocha A.M.A.C., Costa M.F.P., Fernandes E.M.G.P.: A shifted hyperbolic augmented Lagrangian-based artificial fish two-swarm algorithm with guaranteed convergence for constrained global optimization. Engineering Optimization. **48**, 2114–2140 (2016)

[14] Xavier A.E.: Penalização Hiperbólica: Um Novo Método para Resolução de Problemas de Otimização (Ms.c. thesis), Federal University of Rio de Janeiro/COPPE, Rio de Janeiro, Brazil. (1982)

[15] Xavier A.E.: Penalização Hiperbólica (Ph.D. thesis), Federal University of Rio de Janeiro/COPPE, Rio de Janeiro, Brazil. (1992)

[16] Zangwill W.I.: Non-linear programming via penalty functions. Management Sciences. **13**, 344–358 (1967)

[17] Wu H., Luo H., Li S.: The global convergence of augmented Lagrangian methods based on ncp function in constrained nonconvex optimization. Applied Mathematics and Computation . **207**, 124–134 (2009)

[18] Wu H., Luo H., Zheng F., Yang J.: Convergence analysis of modified pth power Lagrangian algorithms with alternative updating strategies for constrained nonconvex optimization. Journal of Computational and Applied Mathematics. **396**, 113608 (2021)