Noname manuscript No.
(will be inserted by the editor)

Dislocation Hyperbolic Augmented Lagrangian Function

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Received: date / Accepted: date

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Abstract In this note, we guarantee the existence of a global saddle point and we study the strong duality for the dislocation hyperbolic augmented Lagrangian function (DHALF). These results are obtained in the context of the convex constrained nonlinear programming.

Keywords Dislocation hyperbolic augmented Lagrangian · Nonlinear programming · Constrained optimization · Convex problem

Mathematics Subject Classification (2000) 90C25 · 90C46

1 Introduction

Throughout this work, we are interested in studying the following optimization problem

$$(P) \qquad x^* \in X^* = argmin\{f(x) \mid x \in S\},$$

where,

$$S = \{x \in I\!R^n \mid g_i(x) \geq 0, \;\; i = 1, ..., m\},$$

is the convex feasible set of the problem (P) and where the function $f : I\!R^n \to I\!R$ is convex and $g_i : I\!R^n \to I\!R,\ i = 1, ..., m$, are concave functions, we assume also that f and g_i are continuously differentiable. Some algorithms of the augmented Lagrangian type that solve the problem (P) are the following [3], [7], [11], [8], [9], [2], [5] and [15]. The existence of saddle points for this problem are studied in [6], [7], [4] and [5].

In this work we are going to introduce the function DHALF, this function is a slight variation of the hyperbolic augmented Lagrangian function (HALF), this function is based on the hyperbolic penalty function (HPF), see [12], [13], [14] and [5]. We will observe that DHALF has similar properties to the Log-Sigmoid Lagrangian (LSL). The function LSL is defined as:

$$L_s(x, \lambda, k) = f(x) - \sum_{i=1}^{m} \frac{\lambda_i}{k_i} \psi(k_i g_i(x))$$

$$= f(x) + \frac{2}{k} \sum_{i=1}^{m} \lambda_i \; ln \left(1 + e^{-k g_i(x)}\right) - \frac{2}{k} \left(\sum_{i=1}^{m} \lambda_i \right) ln(2)$$

and is based on the Log-Sigmoid transformation (LST), this function is defined as $\psi : I\!R \to (-\infty, 2 \; ln(2))$,

$$\psi(t) = 2 \left(ln(2) + t - ln(1 + e^t) \right),$$

and the important features of LST are as follows (for more details of LST and LSL see [9] and [10]):

- $\psi \in C^\infty$ on $(-\infty, \infty)$.
- the LSL is as smooth as the initial functions in the entire primal space.
- ψ' and ψ'' are bounded on $(-\infty, \infty)$.

The contribution of this work is to introduce DHALF, guarantee the existence of a global saddle point for this new function and present some of its properties. We also introduce the dislocation hyperbolic function, this function is a new approach to DHALF.

The work is organized as follows: Chapter 2, some basic results are presented. Chapter 3, DHALF is proposed. In Chapter 4, we guarantee the global existence of saddle point.

2 Preliminaries and Basic Results

Throughout this work, we consider the following assumption:

C1. The optimal set X^* is nonempty, closed, bounded and, consequently, compact.

C2. Slater constraint qualification holds, i.e., there exists $\hat{x} \in S$ which satisfies $g_i(\hat{x}) > 0,\ i = 1, ..., m.$

The Lagrangian function of the problem (P) is $L : I\!R^n \times I\!R^m_+ \to I\!R$, defined as

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x), \tag{2.1}$$

where, $\lambda_i \geq 0,\ i = 1, ..., m$, are called dual variables. The dual function $\Phi : I\!R^m_+ \to I\!R$, is defined as follows

$$\Phi(\lambda) = \inf_{x \in I\!R^n} L(x, \lambda), \tag{2.2}$$

and the dual problem consists of finding

$$(D) \qquad \lambda \in \Lambda^* = argmax\{\Phi(\lambda) \mid \lambda \in I\!R^m_+\}.$$

Since the problem (P) is convex, we know that due to assumption **C2**, the following results will occur: there exists $\lambda^* = (\lambda^*_1, ..., \lambda^*_m)$, such that, the

Karush-Kuhn-Tucker (KKT) conditions hold true, i.e.,

$$\nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0, \tag{2.3}$$

$$\lambda_i^* g_i(x^*) = 0, \quad i = 1, ..., m, \tag{2.4}$$

$$g_i(x^*) \geq 0, \quad i = 1, ..., m, \tag{2.5}$$

$$\lambda_i^* \geq 0, \quad i = 1, ..., m. \tag{2.6}$$

2.1 **Dislocation Hyperbolic Penalty Function**

The dislocation hyperbolic penalty function (DHPF) was proposed in [13] as

$$p(y, \lambda, \tau) = -\lambda y + \sqrt{(\lambda y)^2 + \tau^2} - \tau, \tag{2.7}$$

where $p : (-\infty, +\infty) \times I\!R_+ \times I\!R_{++} \to I\!R$.

We use the following properties of DHPF:

P0) $p(y, \lambda, \tau)$ is $k-$times continuously differentiable for any positive integer k for $\tau > 0$.

P1) $p(0, \lambda, \tau) = 0$, for $\tau > 0$ and $\lambda \geq 0$.

P2) $p(y, \lambda, \tau)$ is decreasing function of y, i.e.,

$$\nabla_y p(y, \lambda, \tau) = -\lambda \left(1 - \frac{\lambda y}{\sqrt{(\lambda y)^2 + \tau^2}} \right) \leq 0,$$

for $\tau > 0$ and $\lambda \geq 0$.

The DHPF is a smoothing of the exact penalty function studied by Zangwill [16].

3 Dislocation Hyperbolic Augmented Lagrangian Function

DHALF is based on DHPF. We define DHALF of problem (P) by l_H : $I\!R^n \times I\!R^m_{++} \times I\!R_{++} \to I\!R$,

$$l_H(x, \lambda, \tau) = f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i, \tau)$$

$$= f(x) + \sum_{i=1}^{m} \left(-\lambda_i g_i(x) + \sqrt{(\lambda_i g_i(x))^2 + \tau^2} - \tau \right), \tag{3.8}$$

where $\tau > 0$ is the penalty parameter. Note that this function belongs to class C^∞ if the involved functions $f(x)$ and $g_i(x),\ i = 1, ..., m$, are too.

Proposition 3.1 *Let us assume that if $f(x)$ and all $g_i(x) \in C^2$ and that $f(x)$ is strictly convex and $g_i(x),\ i = 1, ..., m$, are concave functions, then $l_H(x, \lambda, \tau)$ is strictly convex in $I\!R^n$ for any fixed $\lambda > 0$ and $\tau > 0$.*

Proof. We only need to prove that the Hessiana of l_H is defined positive. Let are $\lambda = (\lambda_1, ..., \lambda_m) > 0$ and $\tau > 0$ fixed. The Hessian of $l_H(x, \lambda, \tau)$ is

$$\nabla^2_{xx} l_H(x, \lambda, \tau) = \nabla^2_{xx} f(x) - \sum_{i=1}^{m} \lambda_i \nabla^2_{xx} g_i(x)$$

$$+ \sum_{i=1}^{m} \left(\frac{(\lambda_i)^2}{\sqrt{(\lambda_i g_i(x))^2 + \tau^2}} - \frac{(\lambda_i)^4 g_i^2(x)}{((\lambda_i g_i(x))^2 + \tau^2)^{\frac{3}{2}}} \right) \nabla_x g_i(x) \nabla_x^T g_i(x)$$

$$+ \sum_{i=1}^{m} \frac{(\lambda_i)^2 g_i(x)}{\sqrt{(\lambda_i g_i(x))^2 + \tau^2}} \nabla^2_{xx} g_i(x). \tag{3.9}$$

What follows is similar to Proposition 4.0.1 of [5]. Therefore, we are going to get $\nabla^2_{xx} l_H(x, \lambda, \tau) > 0$, for $\lambda > 0$ and $\tau > 0$ fixed. ∎

From now on we will consider the following assumption:

C3. For every $\tau > 0$ and $\lambda > 0$, the level set

$$M = \{x \in I\!R^n \ | \ l_H(x, \lambda, \tau) \leq \alpha\},$$

is bounded for every $\alpha < \infty$.

Remark 3.1 *From* **C3** *and Proposition 3.1 for any $\lambda > 0$ and any $\tau > 0$ there exists a unique minimizer*

$$\check{x} = \check{x}(\lambda, \tau) = argmin \{l_H(x, \lambda, \tau) \mid x \in I\!R^n\}$$

for problem (P) with the assumption **C1***.*

4 Duality Theory

The following result is similar to a result obtained in Proposition 4.1.1 of [5] and the Section 7 of [7].

Proposition 4.1 *Consider the convex problem (P). Assume the assumption* **C2** *it hold. Then $x^* \in S$ is a solution of problem (P) for any $\tau > 0$ if and only if:*

(i) There exists a vector $\lambda^* \geq 0$ such that

$$\lambda_i^* g_i(x^*) = 0, \ i = 1, ..., m \quad and \quad l_H(x, \lambda^*, \tau) \geq l_H(x^*, \lambda^*, \tau), \ \ \forall x \in I\!R^n. \tag{4.10}$$

(ii) The pair (x^*, λ^*) is a saddle point of l_H, that is,

$$l_H(x, \lambda^*, \tau) \geq l_H(x^*, \lambda^*, \tau) \geq l_H(x^*, \lambda, \tau), \quad \forall x \in I\!R^n, \ \ \forall \lambda \in I\!R^m_+. \tag{4.11}$$

Proof. (⇒) Let any $\tau > 0$ fixed. Assume x^* is a solution for convex problem (P) satisfying the assumption **C2**. Then system

$$f(x) - f(x^*) < 0,$$

$$-g_i(x) < 0, \; i = 1, ..., m,$$

has no solution in $I\!R^n$. Hence, by the Proper Separation Theorem (see, Theorem 2.26 (iv) of Dhara and Dutta [1]), there exists a vector $(\tilde{\lambda}, \hat{\lambda}) \neq (0, 0) \in I\!R \times I\!R^m$ such that

$$\tilde{\lambda}\,(f(x) - f(x^*)) - \sum_{i=1}^{m} \hat{\lambda}_i g_i(x) \geq 0,$$

for all $x \in I\!R^n$. We rewrite the inequality above as

$$\tilde{\lambda}\,(f(x) - f(x^*)) \geq \sum_{i=1}^{m} \hat{\lambda}_i g_i(x), \tag{4.12}$$

for all $x \in I\!R^n$. Now, we follow an analysis similar to Theorem 4.2 of [1], so by **C2**, we have that there exists $\lambda_i^* = \frac{\hat{\lambda}_i}{\tilde{\lambda}}, \; i = 1, ..., m$, with $\tilde{\lambda} > 0$. Then, by (4.12) we have

$$f(x) - f(x^*) \geq \sum_{i=1}^{m} \lambda_i^* g_i(x), \tag{4.13}$$

for all $x \in I\!R^n$. In particular, (4.13) holds for $x = x^*$. So we get

$$0 \geq \sum_{i=1}^{m} \lambda_i^* g_i(x^*). \tag{4.14}$$

On the other hand, since, $g_i(x^*) \geq 0$ and $\lambda_i^* \geq 0$ for $i = 1, ..., m$, then by (4.14) we obtain

$$\lambda_i^* g_i(x^*) = 0, \;\; i = 1, ..., m, \tag{4.15}$$

holds, so we have the first part of (4.10). Now, we are interested in proving the second part of (4.10). From (4.15) and (4.13), we have

$$f(x^*) - \sum_{i=1}^{m} \lambda_i^* g_i(x^*) = f(x^*) \leq f(x) - \sum_{i=1}^{m} \lambda_i^* g_i(x), \qquad (4.16)$$

for all $x \in I\!R^n$. Now, since we have (4.15), also, we can obtain

$$(\lambda_i^* g_i(x^*))^2 + \tau^2 \leq (\lambda_i^* g_i(x))^2 + \tau^2,\ i = 1, ..., m,$$

so, we have the following

$$\sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2} \leq \sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x))^2 + \tau^2},$$

on both sides we subtract by $\sum_{i=1}^{m} \tau$,

$$\sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2} - \sum_{i=1}^{m} \tau \leq \sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x))^2 + \tau^2} - \sum_{i=1}^{m} \tau, \qquad (4.17)$$

considering (4.16) and (4.17), we have

$$l_H(x, \lambda^*, \tau) \geq l_H(x^*, \lambda^*, \tau), \ \ \forall x \in I\!R^n, \qquad (4.18)$$

in this way, we finish the proof of (4.10). We are interested in verifying item (ii) now. But, first we will prove that $l_H(x^*, \lambda^*, \tau) = f(x^*)$. Indeed, by definition of l_H, we have

$$l_H(x^*, \lambda^*, \tau) = f(x^*) - \sum_{i=1}^{m} \lambda_i^* g_i(x^*) + \sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2} - \sum_{i=1}^{m} \tau, \qquad (4.19)$$

considering (4.15); (4.19) becomes

$$l_H(x^*, \lambda^*, \tau) = f(x^*). \qquad (4.20)$$

On the other hand, as x^* is feasible, i.e.,

$$g_i(x^*) \geq 0, \ \ i = 1, ..., m. \qquad (4.21)$$

By applying the property P2 of the DHF in (4.21), we obtain

$$p(g_i(x^*), \lambda_i, \tau) \leq p(0, \lambda_i, \tau), \quad i = 1, ..., m. \tag{4.22}$$

By applying property P1, on the right side of expression (4.22), we will obtain

$$p(g_i(x^*), \lambda_i, \tau) \leq 0, \quad for \quad \lambda_i \geq 0, \quad i = 1, ..., m. \tag{4.23}$$

By performing the sum of 1 to m in (4.23) it follows that

$$\sum_{i=1}^{m} p(g_i(x^*), \lambda_i, \tau) \leq 0.$$

Adding $f(x^*)$ to both sides of the expression, we obtain

$$f(x^*) + \sum_{i=1}^{m} p(g_i(x^*), \lambda_i, \tau) \leq f(x^*). \tag{4.24}$$

By definition of l_H, (4.24) becomes

$$l_H(x^*, \lambda, \tau) \leq f(x^*). \tag{4.25}$$

Now, by (4.25) and (4.20) we have

$$l_H(x^*, \lambda, \tau) \leq f(x^*) = l_H(x^*, \lambda^*, \tau). \tag{4.26}$$

Finally, from (4.18) and (4.26), there is $\lambda^* \geq 0$ such that the primal-dual solution (x^*, λ^*) is a saddle point of l_H, $\forall x \in I\!R^n$ and $\tau > 0$.

($\Leftarrow$) We assume that (x^*, λ^*) is a saddle point of l_H, so (4.11) is hold. Then, for all $x \in I\!R^n$, $\lambda \in I\!R^m_+$ and for any $\tau > 0$ fixed, we have

$$f(x^*) - \sum_{i=1}^{m} \lambda_i g_i(x^*) + \sum_{i=1}^{m} \sqrt{(\lambda_i g_i(x^*))^2 + \tau^2} - \sum_{i=1}^{m} \tau = l_H(x^*, \lambda, \tau)$$

$$\leq l_H(x^*, \lambda^*, \tau) = f(x^*) - \sum_{i=1}^{m} \lambda_i^* g_i(x^*) + \sum_{i=1}^{m} \sqrt{\left(\lambda_i^* g_i(x^*)\right)^2 + \tau^2} - \sum_{i=1}^{m} \tau. \quad (4.27)$$

From (4.27), we have

$$-\sum_{i=1}^{m} \lambda_i g_i(x^*) + \sum_{i=1}^{m} \sqrt{\left(\lambda_i g_i(x^*)\right)^2 + \tau^2}$$

$$\leq -\sum_{i=1}^{m} \lambda_i^* g_i(x^*) + \sum_{i=1}^{m} \sqrt{\left(\lambda_i^* g_i(x^*)\right)^2 + \tau^2}, \quad (4.28)$$

This relation (4.28) is possible only if $g_i(x^*) \geq 0$. Since, if this relation is violated (*i.e.*, $g_i(x^*) < 0$) for some index i, we can choose λ_i sufficiently large such that (4.28) becames false. So, x^* is feasible for problem (P). We will prove the complementarity condition of (4.10). So again, by (4.28), and since that $\lambda_i \geq 0,\ i = 1, ..., m$, in particular taking $\lambda_i = 0,\ i = 1, ..., m$, in (4.28), we obtain

$$\sum_{i=1}^{m} \tau \leq -\sum_{i=1}^{m} \lambda_i^* g_i(x^*) + \sum_{i=1}^{m} \sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2},$$

thus, it follows that

$$\sum_{i=1}^{m} \left(\lambda_i^* g_i(x^*) + \tau\right)^2 \leq \sum_{i=1}^{m} \left((\lambda_i^* g_i(x^*))^2 + \tau^2\right),$$

$$\sum_{i=1}^{m} \left(\left(\lambda_i^* g_i(x^*)\right)^2 + \tau^2 + 2\tau\lambda_i^* g_i(x^*)\right) \leq \sum_{i=1}^{m} \left((\lambda_i^* g_i(x^*))^2 + \tau^2\right),$$

so,

$$\sum_{i=1}^{m} \lambda_i^* g_i(x^*) \leq 0,$$

and since $\lambda_i^* \geq 0$ and $g_i(x^*) \geq 0,\ i = 1, ..., m$, it must be true

$$\lambda_i^* g_i(x^*) = 0,\ i = 1, ..., m. \quad (4.29)$$

By (4.29) and definition of l_H, we obtain

$$l_H(x^*, \lambda^*, \tau) = f(x^*). \quad (4.30)$$

From definition of saddle point, we know that $l_H(x, \lambda^*, \tau) \geq l_H(x^*, \lambda^*, \tau)$, by (4.30) and by definition of l_H, we can write

$$f(x^*) = l_H(x^*, \lambda^*, \tau) \leq l_H(x, \lambda^*, \tau) = f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i^*, \tau). \qquad (4.31)$$

On the other hand, once again considering property P2 of DHF, for any feasible point x, i.e., $g_i(x) \geq 0,\ i = 1, ...m$, we will carry out a work similar to that of (4.21)-(4.24), thus, we can obtain

$$f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i^*, \tau) \leq f(x), \qquad (4.32)$$

now, we replace (4.32) in (4.31), then follow

$$f(x^*) \leq f(x),$$

therefore, x^* is a global optimal solution of (P). ■

Let's consider the following definitions. Let

$$F_\tau(x) = \sup_{\lambda \geq 0} l_H(x, \lambda, \tau).$$

Then $F_\tau(x) = f(x)$, if $g_i(x) \geq 0,\ i = 1, ..., m$ and $F_\tau(x) = \infty$, otherwise. Therefore, we can consider the following problem

$$x^* = argmin \left\{ F_\tau(x) \mid x \in I\!R^n \right\}, \qquad (4.33)$$

that is the problem (P) reduces to solving (4.33).

Let

$$\phi_\tau(\lambda) = \inf_{x \in I\!R^n} l_H(x, \lambda, \tau)$$

(possibly $\phi_\tau(\lambda) = -\infty$ for some λ) and consider the following dual problem of (P), that consisting of finding

$$\lambda^* = argmax\left\{\phi_\tau(\lambda) \mid \lambda \geq 0\right\}. \tag{4.34}$$

In the following result, we are going to verify the weak duality.

Proposition 4.2 *Let x be a feasible solution to problem (P) and let λ be a feasible solution to problem (4.34). Then*

$$\phi_\tau(\lambda) \leq F_\tau(x) = f(x), \quad \forall x \in S, \ \forall\, \lambda \in I\!R^m_+.$$

Proof. For any feasible x and λ, we then we can get the weak duality. Indeed, by the definition of ϕ_τ, we have

$$\phi_\tau(\lambda) = \inf_{w\in I\!R^n} l_H(w,\lambda,\tau) = \inf_{w\in I\!R^n}\left\{f(w) + \sum_{i=1}^m p(g_i(w),\lambda_i,\tau)\right\}$$

$$\leq \inf_{w\in S}\left\{f(w) + \sum_{i=1}^m p(g_i(w),\lambda_i,\tau)\right\}$$

$$= f(x) + \sum_{i=1}^m p(g_i(x),\lambda_i,\tau). \tag{4.35}$$

Since we know that x is feasible, we have $g_i(x) \geq 0,\ i = 1,...,m$, immediately then, for the property P2 of the HPF, we get the following expressions

$$p(g_i(x),\lambda_i,\tau) \leq p(0,\lambda_i,\tau), \ \ i = 1,...,m,$$

we rewrite the expression above, as follows

$$\sum_{i=1}^m p(g_i(x),\lambda_i,\tau) \leq \sum_{i=1}^m p(0,\lambda_i,\tau),$$

now, we apply property P1, on the right side of the previous inequality

$$\sum_{i=1}^{m} p(g_i(x), \lambda_i, \tau) \leq 0,$$

we add $f(x)$, to both sides of the inequality above

$$f(x) + \sum_{i=1}^{m} p(g_i(x), \lambda_i, \tau) \leq f(x), \tag{4.36}$$

we replace (4.36) in (4.35), so

$$\phi_\tau(\lambda) \leq f(x), \quad \forall x \in S, \ \ \forall \lambda \in I\!R^m_+. \tag{4.37}$$

■

If $\hat{x}$ and $\hat{\lambda}$ are feasible solutions of the primal and dual problems and $F_\tau(\hat{x}) = \phi_\tau(\hat{\lambda})$, then $\hat{x} = x^*$ and $\hat{\lambda} = \lambda^*$. From Remark 3.1, with the smoothness of $f(x)$ and $g_i(x)$, $i = 1, ..., m$, we ensure the smoothness for the dual function $\phi_\tau(\lambda)$.

Theorem 4.1 *The problem (P) is considered. The assumption* **C2** *is verified. Then the existence of a solution of problem (P) implies that the problem (4.34) has a solution and*

$$\phi_\tau(\lambda^*) = f(x^*), \quad for\ any \ \ \tau > 0. \tag{4.38}$$

Proof. Let x^* be a solution of problem (P). By **C2**, we get $\lambda^* \geq 0$, such that (4.10) is verified. So we have

$$\phi_\tau(\lambda^*) = \min_{x \in I\!R^n} l_H(x, \lambda^*, \tau) = l_H(x^*, \lambda^*, \tau)$$

$$\geq l_H(x^*, \lambda, \tau) \geq \min_{x \in I\!R^n} l_H(x, \lambda, \tau) = \phi_\tau(\lambda), \ \ \forall \lambda \geq 0.$$

Therefore $\phi_\tau(\lambda^*) = \max\left\{\phi_\tau(\lambda) \mid \lambda \in I\!R^m_+\right\}$, in this way $\lambda^* \in I\!R^m_+$ is a solution of the dual problem and since we have

$$l_H(x^*, \lambda^*, \tau) = f(x^*),$$

so (4.38) hold. ■

Now we are going to introduce a property of function l_H at a KKT point.

Proposition 4.3 *For any KKT pair* (x^*, λ^*) *and for any* $\tau > 0$.

- $l_H(x^*, \lambda^*, \tau) = f(x^*)$.
- $\nabla_x l_H(x^*, \lambda^*, \tau) = \nabla_x L(x^*, \lambda^*) = \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*) = 0$.
- $\nabla^2_{xx} l_H(x^*, \lambda^*, \tau) = \nabla^2_{xx} L(x^*, \lambda^*) + \frac{1}{\tau} \nabla g(x^*) D^* \nabla g(x^*)$, *where* $D^* = diag\left((\lambda_i^*)^2\right)$, $i = 1, ..., m$.

Proof. Let any $\tau > 0$ fixed.

- by (2.4) we obtain

  $$l_H(x^*, \lambda^*, \tau) = f(x^*) - \sum_{i=1}^m \left(\lambda_i^* g_i(x^*) - \sqrt{\left(\lambda_i^* g_i(x^*)\right)^2 + \tau^2} + \tau\right) = f(x^*).$$

- The gradient of l_H in x is

  $$\nabla_x l_H(x^*, \lambda^*, \tau) = \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \left(1 - \frac{\lambda_i^* g_i(x^*)}{\sqrt{\left(\lambda_i^* g_i(x^*)\right)^2 + \tau^2}}\right) \nabla g_i(x^*),$$

  applying (2.4) we get

  $$\nabla_x l_H(x^*, \lambda^*, \tau) = \nabla f(x^*) - \sum_{i=1}^m \lambda_i^* \nabla g_i(x^*).$$

- The Hessian in x is

$$\nabla^2_{xx} l_H(x^*, \lambda^*, \tau) = \nabla^2_{xx} f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla^2_{xx} g_i(x^*)$$

$$+ \sum_{i=1}^{m} \left(\frac{(\lambda_i^*)^2}{\sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2}} - \frac{(\lambda_i^*)^4 g_i^2(x^*)}{((\lambda_i^* g_i(x^*))^2 + \tau^2)^{\frac{3}{2}}} \right) \nabla_x g_i(x^*) \nabla_x^T g_i(x^*)$$

$$+ \sum_{i=1}^{m} \frac{(\lambda_i^*)^2 g_i(x^*)}{\sqrt{(\lambda_i^* g_i(x^*))^2 + \tau^2}} \nabla^2_{xx} g_i(x^*), \tag{4.39}$$

so, applying (2.4) again we can obtain

$$\nabla^2_{xx} l_H(x^*, \lambda^*, \tau) = \nabla^2_{xx} L(x^*, \lambda^*) + \frac{1}{\tau} \sum_{i=1}^{m} (\lambda_i^*)^2 \nabla g_i(x^*) \nabla g_i(x^*).$$

■

Through this property we can see that the function l_H has the same local property as the function Log-sigmoid Lagrangian (see, [9]) and the logarithmic MBF (see, [7]). Now, we are going to introduce dislocation hyperbolic function, this function is a new approach to DHALF.

Acknowledgements This work was supported by FAPERJ E-26/201.176/2020 (258168) and COPPETEC Foundation, Brazil.

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