A PROXIMAL ALGORITHM WITH VARIABLE METRIC FOR THE P_0 COMPLEMENTARITY PROBLEM

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Abstract

We consider a regularization proximal method with variable metric to solve the nonlinear complementarity problem (NCP) for P_0-functions. We establish global convergence properties when the solution set is non empty and bounded. Furthermore, we prove, without boundedness of the solution set, that the sequence generated by the algorithm is a minimizing sequence for the implicit Lagrangian function, as defined by Mangasarian and Solodov, [18].Those results are stronger than the presented in a previous paper [7].

Key words: Nonlinear complementarity problem, P_0 function, proximal regularization.

1 Introduction

The (NCP) consists of getting $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0.$$

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In this paper we assume that $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a P_0 function, that is, for any x, y in $\mathbb{R}^n$ with $x \neq y$

$$\max_{x_i \neq y_i}(x_i - y_i)(F_i(x) - F_i(y)) \geq 0.$$

We observe that the class of monotone functions is contained in the P_0-class. Applications of (NCP) can be seen in optimization, economics, engineering, and etc, see Harker and Pang, [14].

There are various methods to solve (NCP) for the above class. Facchinei and Kanzow [11] considered the Tikhonov regularization, which consists in solving a sequence of complementarity problems (NCP_k), where the regularized associated function is $F^k(x) := F(x) + c_k x$, c_k being a positive parameter that converges to zero. Yamashita, Imai and Fukushima [27] proposed a proximal regularization, defined as follows: given x^k, let $F^k(x) := F(x)+c_k(x-x^k)$, where c_k is a positive parameter, not necessarily converging to zero. In both methods, if F is a continuously differentiable P_0 function, and the solution set of (NCP) is bounded, they show the global convergence. Da Silva and Oliveira [7] considered a new variable proximal regularization, defined as follows: given $x^k > 0$, let

$$F^k(x) := F(x) + c_k(X^k)^{-r}(x - x^k), \tag{1.1}$$

where $(X^k)^{-r}$ is defined by $(X^k)^{-r} = \mathtt{diag}\{(x_1^k)^{-r}, ..., (x_n^k)^{-r}\}$, $r \geq 1$, c_k being a positive parameter. Da Silva and Oliveira, obtained the global convergence when F is a continuously differentiable P_0 function and the solution set of (NCP) is nonempty and bounded.

Now, some words about the motivation for the regularization (1.1). It comes from the application of some tools of Riemannian geometry to the continuous optimization. This is object of research by many authors, as can be seen in [2, 3, 5, 10, 12, 13, 16, 20, 24, 25], and in the bibliography therein. One of the trends is given in [6], where the association between Riemannian metric dependent gradient and the generator descent direction (also metric dependent) is explored. The authors had unified a large variety of primal methods, seen as gradient ones, and obtained other classes. That includes primal interior point methods such as [8], [17], and the Eggermont multiplicative algorithm, [9]. A general theory for such gradient methods can be seen in [5]. Particularly, in [6], they considered the positive octant R^n_{++} as a manifold, associated with a class of metrics generated by the Hessian of some separable functions $p(x) = \sum_{i=1}^n p_i(x_i)$, $p_i : \mathbb{R}_{++} \rightarrow \mathbb{R}$, $p_i(x_i) > 0$, for $i = 1, 2, ..., n$ (p_i are, at least, C^2 functions). Clearly, the functions defined by $p_i(x_i) = x_i^{-r}$, $i = 1, 2, ..., n$, and $r \geq 1$, are contained in that class. For $r = 1$, $r = 2$, and $r = 3$, they correspond, respectively, to the Hessian of Eggermont multiplicative, log and Fiacco-McCormick barriers. Those metrics, denoted by X^{-r}, lead to projective (affine) interior point methods in [22], and proximal interior point algorithms in [21].

In this paper, we establish global convergence for the algorithm based on the regularization (1.1), when the solution set of the original problem is non empty and bounded, and F is a P_0-continuous function. Besides, without the boundedness hypothesis, we show that the algorithm sequence is minimizing in respect to the implicit Lagrangian function, see [18],

$$\begin{aligned} M(x) &:= \langle x, F(x)\rangle + \\ &+ \frac{\alpha}{2}\left[\left\|\left(x - \frac{1}{\alpha}F(x)\right)_+\right\|^2 - \|x\|^2 + \left\|\left(F(x) - \frac{1}{\alpha}x\right)_+\right\|^2 - \|F(x)\|^2\right], \end{aligned}$$

where $[a_+] = \max\{0, a\}$. We point out that M is a merit function for NCP.

The paper is organized as follows. In section 2, we revise some definitions and results. In section 3, we propose some reformulations for the nonlinear complementarity problem. In section 4, we present a family of regularization functions, studying their properties. Section 5 is dedicated to show the algorithm, and its convergence properties. The paper ends reaching some conclusions.

2 Preliminaries

The concepts that we will present are needed to the development of the following sections. The proofs are refereed.

2.1 Symbols and Notations

We adopt the following notation:

$\mathbb{R}^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, i = 1, \ldots, n\}$ is the vectorial Euclidian space;

Given $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) signifies that $x_i \geq 0$ (respectively, $x_i > 0$) for $i = 1, \ldots, n$;

$\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, \ldots, n\}$;

$\mathbb{R}^n_{++} = \{x = (x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, \ldots, n\}$;

Given $x, y \in \mathbb{R}^n$, $\langle x, y \rangle = \displaystyle\sum_{i=1}^{n} x_i y_i$ is the canonical scalar product in $\mathbb{R}^n$;

Given $x \in \mathbb{R}^n$, $\|x\| = \sqrt{\langle x, x \rangle}$ is the Euclidian norm in $\mathbb{R}^n$;

Given $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, we consider $H(x) = (H_1(x), H_2(x), \ldots, H_n(x))$;

∇ as the gradient of a real function defined in $\mathbb{R}^n$.

Let denote by $[\cdot]$ the orthogonal projection over $\mathbb{R}^n_+$; it is well defined as $\mathbb{R}^n_+$ is a closed convex cone. For $x \in \mathbb{R}^n$ we have $[x]_+ = ([x_1]_+, [x_2]_+, \ldots, [x_n]_+)$, where $[x_i]_+ = \max\{0, x_i\}$. The useful property of non expansiveness is defined by $\|[x]_+ - [y]_+\| \leq \|x - y\|$, for all $x, y \in \mathbb{R}^n$, see, e.g., [15].

2.2 P and P_0-properties

To make clearer the presentation, we put together the following definitions, repeating some of them.

Definition 2.1 *An application $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a*

1. *P_0-function, if, for all $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$\max_{x_i \neq y_i} (x_i - y_i)(H_i(x) - H_i(y)) \geq 0;$$

2. *P-function, if, for all $x, y \in \mathbb{R}^n$, $x \neq y$,*

$$\max_i \{(x_i - y_i)(H_i(x) - H_i(y))\} > 0;$$

3. *Monotone function, if, for all $x, y \in \mathbb{R}^n$,*

$$\langle H(x) - H(y), x - y \rangle \geq 0.$$

Notice that the class of monotone functions is contained in the P_0-function class, any strict monotone function (its definition is obtained by substituting the $\geq$ sign by $>$ in the last one) is a P-function, each P-function is a P_0-function, and P-functions are injective, see, e.g., [19].
The next result will be useful in the proof of the uniqueness of the solution of the regularized problem.

Proposition 2.2 *If $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a P function, then the (NCP) (applied to H), has, at most, one solution.*

Proof. As H is a P-function, it is injective, then, if (NCP) has a solution, it is unique. ∎

The next result will be used in the existence proof of the solution of the regularized problem, see, e. g., [11], [23] e [26].

Lemma 2.3 *Let $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a P_0 continuous function, and $\{u^k\}$ a sequence in $\mathbb{R}^n$ such that $\|u^k\| \to \infty$. Then, there exists a subsequence, which we denote again by $\{u^k\}$, and an index $i \in \{1, 2, \dots, n\}$ such that one of the following conditions occurs:*

(i) *$u_i^k \to \infty$ and the set $\{H_i(u^k)\}$ is lower bounded, or*

(ii) *$u_i^k \to -\infty$ and $\{H_i(u^k)\}$ is upper bounded.*

Proof. Defining the set of indices

$$J := \{j \in \{1, \cdots, n\} : \{u_j^k\} \text{ is unbounded}\},$$

we have that J is non-empty, as $\{u^k\}$ is unbounded. Rearranging the terms, if necessary, we can suppose that $|u_j^k| \longrightarrow \infty$ for all $j \in J$. Define the bounded sequence $\{v^k\}$, given by

$$v_j^k := \left\{ \begin{array}{cc} 0 & \text{if} j \in J \\ u_j^k & \text{if} j \notin J, \end{array} \right. .$$

As H is a P_0-function and $u^k \neq v^k$, there exists an index $i \in J$ such that

$$u_i^k[H_i(u^k) - H_i(v^k)] = (u_i^k - v_i^k)[H_i(u^k) - H_i(v^k)] \geq 0.$$

Notice that, when $|u_i^k| \to \infty$, there are two possible cases to be analyzed. First, suppose that $u_i^k \to \infty$, then, from the above inequality (assuming that $u_i^k > 0$), it follows that $H_i(u^k)$ is lower bounded, as H_i is continuous and $\{v^k\}$ is bounded, so proving part **(i)**. Now, supposing $u_i^k \to -\infty$, the inequality above (assuming that $u_i^k < 0$) shows that $H_i(u^k)$ is upper bounded, thus proving part **(ii)**. ∎

2.3 Banach-Mazur Theorem

Besides the last lemma, the coercivity is also necessary in the proof of existence of the solution of the regularized problem.

Definition 2.4 *$H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a coercive function if, for any sequence $\{u^k\}$, for $\|u^k\| \longrightarrow \infty$, it holds that $\|H(u^k)\| \longrightarrow \infty$.*

The proof of the next lemma is a trivial consequence of above definition.

Lemma 2.5 *Let $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous. Then, the following properties are equivalent:*

(i) *H is coercive;*

(ii) *H is proper, that is, the inverse image $H^{-1}(C)$ of any compact set $C \subset \mathbb{R}^n$ is compact;*

(iii) *the level sets of the application $x \longmapsto \|H(x)\|$ are bounded, i.e., for any constant $\beta \geq 0$, the set $\{x : \|H(x)\| \leq \beta\}$ is bounded.*

The next theorem is a classic result from Banach and Mazur ([4], Theorem 5.1.4). We take it, in the form which is relevant to our using.

Theorem 2.6 *If $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous, locally injective and coercive function, then H is an (upper) homeomorphism of $\mathbb{R}^n$.*

2.4 Exterior Semicontinuity

Definition 2.7 *We say that $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is univalent if it is continuous and injective; and weakly univalent if, there exists some sequence of univalent functions $H_k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, such that H_k converges uniformly to H over each bounded subset of $\mathbb{R}^n$.*

Observe that each P continuous function is univalent. Now, supposing that H is a P_0 continuous function, and defining $H_k(x) := H(x) + \varepsilon_k x$, where $\varepsilon_k > 0$, $\varepsilon_k \longrightarrow 0$ for $k \longrightarrow \infty$, then H_k is a P function for all k, thus, it is injective, and H_k converges uniformly to H over each bounded subset of $\mathbb{R}^n$; it follows that any P_0 continuous function is weakly univalent.

In the proof of convergence of the exact algorithm, we need the following result on exterior semicontinuity, which can be seen in [23].

Theorem 2.8 *Let $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ weakly univalent and suppose that, for some $q^* \in \mathbb{R}^n$, $G^{-1}(q^*)$ is non-empty and compact. Then, for any $\varepsilon > 0$, there exists $\delta > 0$, such that, for weakly univalent functions H and vectors q verifying*

$$\sup_{\overline{\Omega}} \|H(x) - G(x)\| < \delta, \quad \|q - q^*\| < \delta$$

it is true that

$$\emptyset \neq H^{-1}(q) \subseteq G^{-1}(q^*) + \varepsilon \mathcal{B}$$

where $\mathcal{B}$ denotes the open unitary ball in $\mathbb{R}^n$ and $\overline{\Omega}$ is the closure of $\Omega := G^{-1}(q^*) + \varepsilon \mathcal{B}$. Besides, $H^{-1}(q)$ and $G^{-1}(q)$ are non-empty, connected and uniformly bounded, if q is in a neighborhood of q^*.

3 Equivalent Reformulation for (NCP)

In this section, we consider an equivalent reformulation for (NCP), through the minimum function, see [1]. A good reference for other reformulations is the paper of Harker and Pang [14].

With that aim, we define the following class of functions:

Definition 3.1 *We say that $\eta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a NLC-function if it satisfies the following condition:*

$$\eta(a, b) = 0 \iff a \geq 0, \quad b \geq 0 \text{ and } \quad ab = 0. \tag{3.1}$$

Now, defining $\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$\theta(a, b) := \min\{a, b\}, \tag{3.2}$$

we have that θ is a NLC-function. It is a simple task to see that

$$\theta(a, b) = a - [a - b]_+,$$

where $[a]_+ := \max\{a, 0\}$.

At this point, we can define the operator $\Theta : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as

$$\Theta(x) := x - [x - F(x)]_+ = \begin{bmatrix} x_1 - [x_1 - F_1(x)]_+ \\ x_2 - [x_2 - F_2(x)]_+ \\ \vdots \\ x_n - [x_n - F_n(x)]_+ \end{bmatrix}, \tag{3.3}$$

so

$$\Theta(x) = \begin{bmatrix} \theta(x_1, F_1(x)) \\ \theta(x_2, F_2(x)) \\ \vdots \\ \theta(x_n, F_n(x)) \end{bmatrix}. \tag{3.4}$$

The next lemma reformulates the non-linear complementarity problem.

Lemma 3.2 *Let F a function from $\mathbb{R}^n$ in $\mathbb{R}^n$. Then, x^* is a solution of (NCP) if, and only if, x^* is a solution of the system of equation $\Theta(x) = 0$.*

Proof. Supposing that x^* is a solution of (NCP), it holds that $x_i^* \geq 0$, $F_i(x^*) \geq 0$ and $x_i^* F_i(x^*) = 0$ for $i = 1, \ldots, n$. Therefore, using the fact that θ is a NLC-function, (3.4) implies that x^* is a solution of the equation $\Theta(x) = 0$. Reciprocally, admitting that x^* is a solution of $\Theta(x) = 0$, use (3.4) again, to obtain $\theta(x_i^*, F_i(x^*)) = 0$ for $i = 1, \ldots, n$, so, due to the fact that θ is a NLC-function, it follows that x^* is a solution of (NCP). ∎

The next proposition establishes the P and P_0 properties of the operator Θ, defined by (3.4).

Proposition 3.3 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a P function (respectively a P_0-function). Then Θ is a P-function (respectively P_0-function).*

Proof. Assume that F is a P-function. Then, for $x \neq y$, there exists some index i such that

$$(x_i - y_i)[F_i(x) - F_i(y)] > 0.$$

The generality is not lost, if we assume that $x_i > y_i$, then, $F_i(x) > F_i(y)$. We show that $(x_i - y_i)[\Theta_i(x) - \Theta_i(y)] > 0$. Suppose, on the contrary, that this is not true, then $\Theta_i(x) \leq \Theta_i(y)$, which means that

$$x_i - [x_i - F_i(x)]_+ \leq y_i - [y_i - F_i(y)]_+. \tag{3.5}$$

Now, we are going to show that above inequality does not occur. The possible values for $x_i - [x_i - F_i(x)]_+$ are

(1) x_i, if $x_i - F_i(x) \leq 0$, and

(2) $F_i(x)$, if $x_i - F_i(x) > 0$;

and, the possible values for $y_i - [y_i - F_i(y)]_+$ are

(a) y_i, if $y_i - F_i(y) \leq 0$, and

(b) $F_i(y)$, if $y_i - F_i(y) > 0$.

Now, considering all possible values above, we will show that (3.5) will be false.

(i) Suppose (1) and (a), then (3.5) implies $x_i \leq y_i$, which is a contradiction.

(ii) Suppose (1) and (b), then (3.5) implies $x_i \leq F_i(y)$. As $x_i > y_i$, that implies $y_i < F_i(y)$, which contradicts the condition in (b).

(iii) Supposing (2) and (a), we have, from (3.5) that $F_i(x) \leq y_i$. Now, (a) leads to $y_i \leq F_i(y)$, which implies that $F_i(x) \leq F_i(y)$, which is a contradiction.

(iv) Finally, supposing (2) and (b), (3.5) furnishes $F_i(x) \leq F_i(y)$, that is a contradiction.

Then, for any $x \neq y$ in $\mathbb{R}^n$, there exists some index i such that

$$(x_i - y_i)[\Theta_i(x) - \Theta_i(y)] > 0,$$

proving that Θ is a P-function. A similar argument shows that Θ is a P_0-function. ∎

From that proof, we see that, if F is a P-function, and $x \neq y$, there exists some index i such that

$$(x_i - y_i)[F_i(x) - F_i(y)] > 0.$$

Therefore, for the same index i, we have

$$(x_i - y_i)[\Theta_i(x) - \Theta_i(y)] > 0.$$

4 Variable Proximal Regularization

In this section, we present the proximal regularization, applied to a function $F : \mathbb{R}^n \to \mathbb{R}^n$. Analogously to the previous section, we will define the corresponding operator associated to the regularized function, studying its properties.

4.1 Regularization of F

Given $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $x^k \in \mathbb{R}^n$, $x^k > 0$, $r \in \mathbb{R}$, $r \geq 1$, the regularization of F, $F^k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, is defined by

$$F^k(x) := F(x) + c_k(X^k)^{-r}(x - x^k), \tag{4.1}$$

where $(X^k)^{-r}$ means $(X^k)^{-r} = \mathtt{diag}\{(x_1^k)^{-r}, \ldots, (x_n^k)^{-r}\}$, c_k being a positive parameter.

The next result shows, using Proposition 2.2, item **(c)**, that, for P_0-functions, the regularized problem (NCP_k), has, at most, one solution, thus it is more tractable then the original (NCP).

Proposition 4.1 *If F is a P_0-function, then F^k is a P-function.*

Proof. As F is a P_0-function, given $x \neq y$ in $\mathbb{R}^n$, there exists an index i such that

$$(x_i - y_i)[F_i(x) - F_i(y)] \geq 0. \tag{4.2}$$

We will show that $(x_i - y_i)[F_i^k(x) - F_i^k(y)] > 0$, thus proving that F^k is a P-function. Using the definition of F^k, it holds that

$$\begin{aligned}
(x_i - y_i)[F_i^k(x) - F_i^k(y)] &= (x_i - y_i)(F_i(x) - F_i(y)) + \\
&+ c_k(x_i - y_i)[\sum_{j=1}^{n}(X^k)_{ij}^{-r}(x_j - y_j)] \\
&= (x_i - y_i)(F_i(x) - F_i(y)) + c_k(x_i^k)^{-r}(x_i - y_i)^2.
\end{aligned}$$

Now, using (4.2) and the facts $x_i \neq y_i$, $x_i^k > 0$ and $c_k > 0$, we achieve $(x_i - y_i)[F_i^k(x) - F_i^k(y)] > 0$. ∎

4.2 Minimum Operator Regularization

Let the regularization of F, $F^k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, furnished in (4.1). We define, similarly to (3.3), the operator $\Theta^k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, as

$$\Theta^k(x) := x - [x - F^k(x)]_+ = \begin{bmatrix} \theta(x_1, F_1^k(x)) \\ \theta(x_2, F_2^k(x)) \\ \vdots \\ \theta(x_n, F_n^k(x)) \end{bmatrix}, \tag{4.3}$$

where $\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given in (3.2).

We, also, define the merit function corresponding to $\psi^k : \mathbb{R}^n \longrightarrow \mathbb{R}$ as

$$\psi^k(x) = \frac{1}{2}\|\Theta^k(x)\|^2. \tag{4.4}$$

The next result is a direct consequence of Proposition 3.3.

Proposition 4.2 *Let F a P_0-function, then Θ^k is a P-function.*

Proof. From the Proposition 4.1, we see that F^k is a P-function. Therefore, it follows from Proposition 3.3, that Θ^k is a P-function. ∎

Now, we are going to prove the existence of solution of the regularized problem (NCP_k).

Proposition 4.3 *Let F a P_0-continuous function, then Θ^k is coercive.*

Proof. Let $\{u^j\}$ a sequence verifying $\|u^j\| \longrightarrow \infty$. As F is a P_0-continuous function, the Lemma 2.3 implies the existence of a subsequence, whose notation we maintain as $\{u^j\}$, and an index i such that one of the following conditions occurs:

(i) $u_i^j \to \infty$ and $\{F_i(u^j)\}$ is lower bounded, or,

(ii) $u_i^j \to -\infty$ and $\{F_i(u^j)\}$ is upper bounded.

From F^k definition, it is true that

$$\theta(u_i^j, F_i^k(u^j)) = u_i^j - [u_i^j - F_i(u^j) - c_k(x_i^k)^{-r}(u_i^j - x_i^k)]_+. \tag{4.5}$$

Now, supposing (i), we are going to show that, in the equality above,

$$\theta(u_i^j, F_i^k(u^j)) \longrightarrow \infty.$$

There are two cases to consider:

(a) If $u_i^j - F_i(u^j) - c_k(x_i^k)^{-r}(u_i^j - x_i) \leq 0$, then, from (4.5), we have $\theta(u_i^j, F_i^k(u^j)) = u_i^j$, so, $\theta(u_i^j, F_i^k(u^j)) \longrightarrow \infty$;

(b) If $u_i^j - F_i(u^j) - c_k(x_i^k)^{-r}(u_i^j - x_i^k) > 0$, then, from (4.5), it gets

$$\theta(u_i^j, F_i^k(u^j)) = F_i(u^j) + c_k(x_i^k)^{-r}u_i^j - c_k(x_i^k)^{-r}x_i^k.$$

As $\{F_i(u^j)\}$ is lower bounded and $c_k(x_i^k)^{-r}x_i^k$ is a constant, we obtain $\theta(u_i^j, F_i^k(u^j)) \longrightarrow \infty$.

As an outcome, we see that occurring (i), leads to $\|\Theta^k(u^j)\| \longrightarrow \infty$. The proof is analogous, if we have (ii). ■

We finish this section with the uniqueness theorem for the solution of (NCP_k), when F is a continuous P_0-function.

Theorem 4.4 *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ a P_0-continuous function, and $F^k : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as in* (4.1)*, then the equation system $\Theta^k(x) = 0$ has an unique solution.*

Proof. Use the fact that F is a P_0-function and Proposition 4.2 to see that Θ^k is a P-function, so injective. Additionally, from Proposition 4.3, Θ^k is coercive. At this point, we can apply Theorem 2.6, obtaining that Θ^k is an homeomorphism of $\mathbb{R}^n$, then the solution of $\Theta^k(x) = 0$ is unique. ■

Observe that, due to Lemma 3.2 and the above theorem, it follows that the solution of (NCP_k) is unique.

5 Algorithm and Convergence

The minimum operator, given in (4.3), will be used in the formulation of an exact proximal algorithm, with variable metric, to solve (NCP), where $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a continuous P_0-function [1]. We prove convergence properties, under such hypothesis as the boundedness of the (NCP) solution set. Also, without this condition, we show that the (exact) sequence is minimizing for the implicit Lagrangian function, $M : \mathbb{R}^n \longrightarrow \mathbb{R}_+$, defined by

[1] We thanks to Prof. M. S. Gowda who suggested the only using of the continuity of F.

$$M(x) := \langle x, F(x)\rangle + \frac{\alpha}{2}(\|[x - \frac{1}{\alpha}F(x)]_+\|^2 - \|x\|^2 + \|[F(x) - \frac{1}{\alpha}x]_+\|^2 - \|F(x)\|^2), \tag{5.1}$$

$0 < \alpha < 1$ being a given scalar. It was introduced by Mangasarian and Solodov [18], having the useful merit function property for the (NCP).

In the following, we present the exact algorithm, where ψ^k in (4.4) is the merit function.

Algorithm 1 **Step 0:** *Choose* $c_0 > 0$, $\delta_0 \in (0, 1)$ *and* $x^0 \in \mathbb{R}^n_{++}$. *Set* $k := 0$.

Step 1: *Given* $c_k > 0$, $\delta_k \in (0, 1)$ *and* $x^k \in \mathbb{R}^n_{++}$, *compute* $x^{k+1} \in \mathbb{R}^n_{++}$, $\beta_{k+1} \in (0, 1)$ *such that* $\beta_{k+1} < \beta_k$, $\|\overline{x}^k - x^{k+1}\| \le \beta_{k+1}$ *and* $\psi^k(x^{k+1})^{\frac{1}{2}} \le \delta_k$, *where* $\overline{x}^k$ *is the unique solution of* (NCP_k).

Step 2: *Choose* $c_{k+1} > 0$ *and* $\delta_{k+1} \in (0, \delta_k)$. *Set* $k := k + 1$ *and go back to Step 1.*

The Algorithm 1 is well defined. Indeed, due to Theorem 4.4, the (NCP_k) has an unique solution, besides, as ψ^k is continuous, it is true that, for any $\delta_k > 0$, there exists some $0 < \beta_{k+1} < \beta_k$ such that $\psi^k(y)^{\frac{1}{2}} \le \delta_k$, for any y satisfying $\|\overline{x}^k - y\| \le \beta_{k+1}$, as $\psi^k(\overline{x}^k) = 0$.

Now, the global convergence will be ensured if we let the following hypothesis about $\{c_k\}$:

(A) $c_k(X^k)^{-r}(x^{k+1} - x^k) \longrightarrow 0$ if $\{x^k\}$ is bounded;

(B) $c_k(X^k)^s \longrightarrow 0$ if $\{x^k\}$ is unbounded and $s \le 0$;

Observe that hypothesis **B** does not necessarily imply that c_k goes to zero, as in the proximal regularization analyzed by Yamashita, Imai and Fukushima, in [27].

Now, we return to the functions aggregated to the minimum functions, given by Θ (3.3), Θ^k (4.3) and ψ^k (4.4). Also, define $\psi(x) = \frac{1}{2}\|\Theta(x)\|^2$. Then, through the above hypothesis, ψ^k will converge uniformly to ψ in compacts.

Lemma 5.1 *Suppose that hypothesis* **B** *is true, and* $S \subset \mathbb{R}^n$ *is a non empty compact set. Additionally, let* $\{x^k\}$ *unbounded. Then, for each* $\varepsilon > 0$, *there exists some sufficiently large* k_0, *such that:*

(i)

$$\|\Theta^k(x) - \Theta(x)\| < \varepsilon \quad para\ todo \quad x \in S;$$

(ii)

$$|\psi^k(x) - \psi(x)| < \varepsilon \quad para\ todo \quad x \in S.$$

for $k \geq k_0$.

Proof. (i) From the definition of Θ^k and the non-expansivity of the projection (see [15], pp. 118), we have, for all $x \in S$, that

$$\begin{aligned} \|\Theta^k(x) - \Theta(x)\| &= \|[x - F(x)]_+ - [x - (F(x) + c_k(X^k)^{-r}(x - x^k))]_+\| \\ &\leq \|c_k(X^k)^{-r}(x - x^k)\| \leq \|c_k(X^k)^{-r}\|\|x\| + \|c_k(X^k)^{1-r}\| \end{aligned}$$

Now, taking the limit $k \longrightarrow \infty$ in above inequality, and using hypothesis **B**, we get the uniform convergence of Θ^k to Θ in S. **(ii)** Utilize the definitions of ψ^k and ψ to get, for all $x \in S$

$$\begin{aligned} |\psi^k(x) - \psi(x)| &= \frac{1}{2}|\|\Theta^k(x)\|^2 - \|\Theta(x)\|^2| \\ &= \frac{1}{2}|(\|\Theta^k(x)\| - \|\Theta(x)\|)(\|\Theta^k(x)\| + \|\Theta(x)\|)| \\ &\leq \frac{1}{2}(\|\Theta^k(x)\| + \|\Theta(x)\|)(\|\Theta^k(x) - \Theta(x)\|). \end{aligned}$$

As above, take $k \longrightarrow \infty$ in the inequality, and use part **(i)**, to obtain that ψ^k converges uniformly to ψ in S. ■

In the next theorem, we obtain the global convergence of the algorithm.

Theorem 5.2 *Given F, a P_0-continuous function, suppose valid the hypothesis* **A** *and* **B**, *and that the (NCP) solution set, denoted by S^*, is non empty and bounded. Letting $\delta_k \longrightarrow 0$, then the sequence $\{x^k\}$, generated by the algorithm 1 is bounded, and any cluster point of $\{x^k\}$ is a solution of (NCP).*

Proof. Let show that $\{x^k\}$ is bounded. Admit that $\{x^k\}$ is unbounded, then, there exists some subsequence $\{x^{k_j}\}$, such that $\left\|x^{k_j}\right\| \to \infty$. As S^* is bounded, by hypothesis, we have, as a consequence, that, for any $\delta > 0$, $S := S^* + \delta\mathcal{B}$, is non-empty, bounded, and $x^{k_j} \notin S$ for sufficiently large

j. We can apply Lemma 5.1, to obtain that, for such j, Θ^{k_j-1} is close to Θ, in the closure $\overline{S}$ of S. Therefore, using Theorem 2.8, it follows that $(\Theta^{k_j-1})^{-1}(0) \subseteq (\Theta)^{-1}(0) + \delta\mathcal{B} = S$. However $\overline{x}^{k_j-1} = (\Theta^{k_j-1})^{-1}(0)$, so $\overline{x}^{k_j-1} \in S$, which is a contradiction. Now, in order to show the cluster point property, we first observe that as $\{x^k\}$ is bounded, the hypothesis **A** and the definition of ψ^k lead to $|\psi^k(x^{k+1}) - \psi(x^{k+1})| \longrightarrow 0$. Therefore, from the Step 1 of the algorithm, and the hypothesis $\delta_k \longrightarrow 0$, we get $\phi^k(x^{k+1}) \longrightarrow 0$. Clearly, $\phi(x^{k+1}) \longrightarrow 0$, meaning that any cluster point is a solution of (NCP). ■

The next result is an immediate consequence of the algorithm and above theorem.

Corollary 5.3 *Let F a P_0-continuous function, suppose valid the hypothesis* **A** *and* **B***, and that the (NCP) solution set S^*, is non-empty and bounded. Additionally, take $\delta_k \longrightarrow 0$ and choose $\{\beta_k\}$, such that $\beta_k \longrightarrow 0$. Then, any cluster point of $\{\overline{x}^k\}$ is a solution of (NCP).*

The next theorem presents the result that $\overline{x}^k$ is a minimizing sequence for the implicit Lagrangian function (5.1), without using the unboundedness condition over the solution set.

Theorem 5.4 *Suppose that F is a P_0-continuous function and the parameter sequence $\{c_k\}$ verifies*

$$\lim_{k\to\infty} c_k (X^k)^{-r}(\overline{x}^k - x^k) = 0. \tag{5.2}$$

Then, $\{\overline{x}^k\}$ is a minimizing sequence relative to the implicit Lagrangian function, given in (5.1), that is,

$$\lim_{k\to\infty} M(\overline{x}^k) = 0.$$

Proof. We start, by showing the following inequality, valid for M, see [28],

$$M(x) \leq \frac{1-\alpha^2}{2\alpha}\|x - y_\alpha(x)\|^2, \tag{5.3}$$

where

$$y_\alpha(x) := [x - \frac{1}{\alpha}F(x)]_+.$$

As $\overline{x}^k$ is the unique solution for (NCP_k), Lemma 3.2 says that

$$\overline{x}^k = [\overline{x}^k - \frac{1}{\alpha} F^k(\overline{x}^k)]_+.$$

Therefore, using also the non-expansivity of the projection operator, it holds that

$$\begin{aligned} \|\overline{x}^k - y_\alpha(\overline{x}^k)\| &= \|[\overline{x}^k - \frac{1}{\alpha} F^k(\overline{x}^k)]_+ - [\overline{x}^k - \frac{1}{\alpha} F(\overline{x}^k)]_+\| && (5.4) \\ &\le \frac{1}{\alpha} \|c_k (X^k)^{-r} (\overline{x}^k - x^k)\|. && (5.5) \end{aligned}$$

Now, apply (5.2), (5.3) and (5.4) to get

$$\lim_{k\to\infty} M(\overline{x}^k) \le \frac{1-\alpha^2}{2\alpha^3} \lim_{k\to\infty} \|c_k (X^k)^{-r} (\overline{x}^k - x^k)\|^2 = 0.$$

Finally, recalling that, as a merit function for the NCP, $M(x) \ge 0$, it follows that $\{\overline{x}^k\}$ is minimizing for the implicit Lagrangian function. ∎

Conclusion

We presented new convergence results on a class of algorithms to solve the (NCP), for P_0-functions. For the algorithm 1, which is, in a certain sense, exact, we have shown the global convergence, only supposing the continuity of F, and the minimizing sequence property for a certain merit function, without supposing the boundedness of the solution set of the (NCP). As future work, in are interested on the dependence of the $r \ge 1$ parameter, and the application to monotone functions. With respect to the computation of the sequence c_k, other choices can be considered, taking in account the necessary theoretical changes, due to the using of the minimum function. Finally, as an open problem, it is to be considered the global convergence of the algorithm, without the boundedness of the solution set of NCP.

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