On b-perfect graphs*

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Abstract

The *b-chromatic number* $\varphi(G)$ of a graph G is defined as the largest number k for which the vertices of G can be colored with k colors satisfying the following property $\mathcal{P}$: for each $i,\ 1 \leq i \leq k$, there exists a vertex x_i of color i such that for all $j \neq i,\ 1 \leq j \leq k$ there exists a vertex y_j of color j adjacent to x_i. A graph G is $b\omega$-perfect if $\varphi(H) = \omega(H)$ for every induced subgraph H of G. We prove that every P_4-free graph is $b\omega$-perfect if and only if it is $2D$-free and $3P_3$-free.

Keywords : b-chromatic number, P_4-free graph, $b\omega$-perfect.

1 Introduction

Parameters involving vertices or edges coloring have attracted a lot of attention and have been extensively studied [5]. The interest in these parameters comes mostly from the algorithmic graph theory.

In this paper, we define a $k-$coloring of G as a function c defined on $V(G)$ into a set of colors $C = \{1, 2, \cdots k\}$ such that any two adjacent vertices have different colors. The term *proper coloring* is sometimes used when one wants to insist on the condition $c(x) \neq c(y)$ for all $xy \in E(G)$. The minimum cardinality k for which G has a $k-$coloring is the *chromatic number* $\chi(G)$ of G. It is well known that determining the chromatic number of a graph is NP-hard for general graphs, but polynomial-time solvable for certain classes of graphs [3, 6]. For

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instance, a graph has chromatic number 2 if and only if it is bipartite. Earlier, in 1941 Brooks [2] proved that: $\chi(G) \leq \Delta(G) + 1$.

In this paper, we are interested in the *b-chromatic number* $\varphi(G)$ defined as the largest number k for which the vertices of G are colored with k colors satisfying the following property $\mathcal{P}:$ for each i, $1 \leq i \leq k$, there exists a vertex x_i of color i such that for all $j \neq i$, $1 \leq j \leq k$ there exists a vertex y_j of color j adjacent to x_i. Each vertex x_i is said to be φ-dominant.

The b-chromatic number was first defined and studied by Irving and Manlove [4]. They showed that determining $\varphi(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees.

A graph G is $b\omega$-**perfect** if $\varphi(H) = \omega(H)$ for every induced subgraph H of G. A graph is $b\omega$-**minimal imperfect** if it is not $b\omega$-perfect and all its induced subgraphs are $b\omega$-perfect. Let $b = \varphi(G)$.

The aim of this work is to prove the following result.

Theorem 1 *Let G be a P_4-free graph, then we have the equivalence:*

(i) *G is $b\omega$-perfect*

(ii) *G is $2D$-free and $3P_3$-free.*

It is obvious that if G contains $2D$ or $3P_3$ then G is not $b\omega$-perfect.

The proof of " (ii) implies (i)" will be done by contradiction. Consider G, a P_4-free $2D$ and $3P_3$ free, $b\omega$-minimal imperfect graph.

We observe that no component of G is a clique, otherwise it contains no φ-dominant vertex, and we will have a contradiction with the minimality of G.

From now one each vertex φ-dominant will be said dominant.

2 Definitions and preliminary results

We consider simple non oriented simple graphs. In a graph G, we denote by $N(x)$ the neighborhood of a vertex x, by ω the order of a maximum clique of G. Any clique considered here is a maximum clique. A graph is P_k-free if it contains no induced path with k vertices (P_k). A graph is diamond free or D-free if it contains no diamond: a complete graph with four vertices minus an edge ($K_4 - e$). It is known [1] that in any connected graph P_4-free, any maximum clique is dominating. Furthermore, if the graph is not 2-connected but connected, then it has exactly one cutvertext x_0, and this cutvertex dominates the graph.

We shall prove the following results.

Proposition 2.1 *If H is a P_4-free 2-edge connected graph, not a clique and $\omega(H) \geq 3$, then H contains a diamond.*

Proof: It is sufficient to consider a maximum clique and a vertex y outside this clique. We know that there exists at least an edge $[y, c]$ with c vertex of the clique. There is a vertex u of the clique such that y and u are independent. As H is 2-edge connected, there exists a path $P(y, a)$ (with $a \neq c$) from y to the clique. As H is P_4-free, we have a diamond containg the triangle (u, a, c). □

It follows that: If H is a P_4-free, imperfect graph and $\omega(H) \geq 3$, then H contains a diamond. Furthermore if H is not 2-connected then it contains at least 2 diamonds ($2D$).

Observations:

1. In a connected graph, but not 2 connected, P_4-free, there is a unique cutvertex and this cutvertex is a universal vertex, so it is in every clique.
2. In any minimal (connected or not) imperfect graph, any vertex t which is outside a clique K is either the unique neighbour of colour $c(t)$ of some dominant or the unique dominant of colour $c(t)$.

From now on we consider minimal imperfect graphs of minimal order. We remark that any dominant vertex is a center of a P_3.

It follows that if G is not connected, it is $3P_3$ free and minimal imperfect, then:

remark 1 G has most 2 components G_1 and G_2; we may suppose that $\omega(G_1) = \omega(G)$.

remark 2 At most ω dominant vertices of different colours of G are in G_1 and at least one is in G_2:

Indeed: if we have $p \geq \omega + 1$ dominant vertices x_1 ,..., x_p of different colours of G in G_1, we get a p-dominating colouring of G_1 by coloring the non dominant vertices z_i of a colour $i \geq p+1$ by a colour in $\{1, ..., p\}$ missing in $N(z_i)$. We get a p dominant coloring of G_1 which is a contradiction with the minimality of G as $\omega(G_1) < p$.

remark 3 For any maximal clique in G_1, as G is P_4-free, the clique dominates G_1, so at least one vertex of the clique is φ dominant. As $\varphi(G) > \omega$ and G is P_4-free, at least one vertex of this clique is not dominant.

remark 4 No dominant vertex y_i, of colour i, which is outside a clique, is adjacent to a non dominant vertex u contained in the clique. Otherwise if b is a colour external to the clique, either $i = b$ it is obvious, or y_i is adjacent to a vertex r_b and $\{r_b, y_i, u, u_i\}$ induces a P_4, where u_i is a vertex of K of colour i.

Lemma 2.2 *Let G be a graph P_4-free. Let $P = (a, b, c)$ and $P' = (d, e, f)$ be two disjoint induced paths of length 2 in G with independent centers b and c. If the vertices a, b are independent from d, then the sets $\{a, b, c\}$ and $\{d, e, f\}$ are relatively independent.*

Lemma 2.3 *If G is bω-minimal imperfect, then $\varphi(G) = \omega + 1$, and for each colour j, there exists a clique $K_{\hat{j}}$.*

Proof: Let i be any color. Let C_i be the set of vertices of colour i. We have $\omega(G) \leq \varphi(G) - 1 \leq \varphi(G - C_i)$, then by minimality of G, $\varphi(G - C_i) = \omega(G - C_i) \leq \omega(G)$. It follows that we get the equalities
$\omega(G) = \varphi(G) - 1 = \varphi(G - C_i)$ □

We may suppose that the colours are $\{1, 2, ..., b\}$ with $b = \omega + 1$.

Corollary 2 *Let G be bω-minimal imperfect and P_4-free, let ω_1 be the maximum number of dominant vertices contained in a clique of G,* *let $K = K_{\widehat{b}}$ be a clique containing ω_1 dominant vertices and let $\mathcal{D}$ be a set of dominant vertices of different colours contained in $G - K$. Then we have:*

1. *$1 \leq \omega_1 \leq \omega - 1$. The number of nondominant vertices in $G - K$ is at least $\omega(\omega - \omega_1)$,*
2. *$\mathcal{D}$ is a stable set.*

Proof: We may suppose $K \subset G_1$.
Proof of 1)

We know by remark 3, that $\omega_1 \neq 0$. If $\omega_1 = \omega$, each vertex of K has a neighbour outside K of colour b. By maximality of the order of the clique, there does not exist a common neighbour to all the vertices of K. So by minimality of the graph G, we have at least two vertices u_b and u'_b, of colour b outside K, with priviliged neighbour,respectively, x_1 and x_2 in K; and $\{u_b, x_1, x_2, u'_b\}$ induces a P_4. We have a contradiction. Then $1 \leq \omega_1 \leq \omega - 1$. By remark (4), a non dominant vertex is contained into at most one clique. By Lemma 2.3, there are at least $(\omega + 1)$ cliques, so we have at least $\omega(\omega - \omega_1)$ non dominant vertices outside K.

Proof of 2)

Case G not connected: $\mathcal{D}$ is composed by 2 sets W_2 and W'_2. Let W_2 be a maximal set of dominating vertices of different colours contained in G_1, let W'_2 be a set of dominating vertices of all the colours with no dominant vertex in G_1, one vertex by colour.

Let $w_2 = |W_2|$, Let $w'_2 = |W'_2|$.

• there is at most one vertex of colour b, which is neighbour of the ω_1 dominant vertices contained in K.

•• Each $y \in W_2$ needs at most $\omega - 1$ neighbours outside K; each vertex of W'_2 needs at most ω neighbours. Suppose that two vertices y_i and y_j ,of W_2 or W'_2 are adjacent (as G is P_4-free).

i)if y_i and y_j are in W_2,

the two vertices y_i and y_j need in common at most $\omega - 2$ colours, so the number of the non dominant outside K is at most :

$$\omega(\omega - \omega_1) - \omega_2 + 1$$

.

ii) if y_i and y_j are in W_2", they need at most $\omega - 2$ neighbours. So the number of the non dominant outside K is at most :
$\omega(\omega - \omega_1) - \omega_2$. In case (i) $\omega_2 \geq 2$. By (1), this case cannot holds. In the second case, by (1), we have $\omega_2 = 0$, and there is exactly one edge in $\mathcal{D}$ (α). Either $\omega_1 \geq 2$, we have a $2D_2$: one containing two dominant vertices contained in K, and the other one containing y_i y_j; or $\omega_1 = 1$,by maximality of K, as the 2 dominant vertices are in a triangle, then necessarily $\omega \geq 4$, $\omega - \omega_1 + 1 \geq 4$. At least three dominant vertices are in W_2'. We have a P_3 in G_1, and by (α), we have a $2P_3$ with centers in $\mathcal{D}$.

Case G connected: Two vertices of $\mathcal{D}$ which are adjacent need at most $(\omega - 2)$ neighbours together. So outside K we have at most
$(\omega - 1)(\omega - \omega_1 - 1) + (\omega - 2)$ non dominant vertices; which is at most $(\omega - 1)(\omega - \omega_1)$. We have a contradiction with (i).

3 Proof of the theorem

We use the notations of the precedent section. Let n' be the number of non dominant vertices outside K. We consider the set $\mathcal{D}$. Let $c(x)$ denote the colour of x. Let r be the number of common neighbours of at least 2 vertices of $\mathcal{D}$. Then

$$n' \leq \omega.(\omega - \omega_1 + 1) - \omega_2 - r + 1.$$

So by (ii) of the precedent corollary, $r \leq \omega - \omega_2 - 1$. So, each vertex y_i of $\mathcal{D}$, either in G_1 or in G_2, has at least one priviliged neighbour, $r(i)$, outside K. Either $c(r_i) \neq b$, and as there is a vertex u of colour $c(r_i)$ in K, or $c(r_i) = b$ and there exists a vertex u of colour $c(y_i)$, not dominant, in K. Then u is independent from r_i. As G is without P_4, it follows that $N(y_i) \cap K \subset N(r_i)$. As $r(i)$ is not dominant by (ii) and G is P_4-free, then there exists necessarily outside K, a neighbour $r'(i) \notin$ of y_i which is independent from $r(i)$.

If $\omega - \omega_1 \geq 2$, there are at least three dominant vertices outside K. By applying two times Lemma 2.2, we see that we have at least three paths of length 2 which are independent. If $\omega - \omega_1 = 1$, by maximality of ω_1 each vertex outside K is independent from at least a dominant vertex contained in K. As G is P_4-free, there exists at least a dominant vertex $x_k \in K$ which is independent from y_1 and $y_b \in \mathcal{D}$. With the path $[u_i, x_k, u_b\ ]$, and by Lemma 2.2, we get three independent paths centered in $\mathcal{D} \cup x_k$. This proofs the theorem. $\diamond$

References

[1] G. Bacsó and Z. S. Tuza, Dominating cliques in P_5-free graphs, *periodica Mathematica Hungarica*,**21**(4), (1990), pp 303-308.

[2] R. L. Brooks, On colouring the nodes of a network, *Proc. Cambridge Plolos. Soc.*, **37**,(1941), pp 194-197.

[3] M. R. Garey, D. S. Johnson, Computers and Intractability, Freeman, San Francisco, CA, (1979).

[4] R. W. Irving and D. F. Manlove, The b-chromatic number of a graph, *Discrete Applied Mathematics*, **91**, (1999), pp 127-141.

[5] T. R. Jensen and B. Toft, Graph Coloring Problems, Wiley Interscience Publication Series in *Discrete Mathematics and Optimization*, 1995.

[6] R. M. Karp, Reducibility among combinatorial problems, Complexity os Computer Computations, in R.E. miller and J. W. Thacher (eds), Plenun Press, New York (1972), pp 85-103.