Epistemic Modal Tableaux with Common Knowledge

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Abstract. This article describes a tableaux method for epistemic modal logic. It was designed to deal with multi-agent systems and is an extension and generalization of the method presented in [CFGH1998] to classic modal logic. Both are pattern-driven methods and do not use tree as basic structures. In our case, the structure is a multi rooted directed acyclic graphs. In particular, this method can handle knowledge and common knowledge operators [FHMV2000]. No restriction is made on the number of relations (each representing an agent). Furthermore the relations do not need to have the same set of relational properties: the set of relational properties of each relation can hold any combination of the systems T, 4, B, 5, D, De and C plus the K system.

1 Introdution

Epistemology is the study of knowledge. It is also an important field of logic as it is about modelling and reasoning about knowledge and belief in multi-agent systems. Therefore it has applications in artificial intelligence, computer science and other fields. The analysis of a conversation, a bargain section, a communication protocol in a distributed system can be modelled as a multi-agent system. An agent in a multi-agent system, or group, must not only take into account facts about the environment, but also about knowledge (or epistemic state) of others agents. The epistemic state of a group of agents can be modelled as a Kripke model (or structure). Such representation relates the notions of possible worlds and properties of "knowing a fact". In a Kripke structure each agent has a binary relation. We say that an agent "knows" a fact in a particular world w if that fact holds in all possible worlds accessible from w. Another important notion is the common knowledge operators and their negation. Common knowledge is an important "condition" or "pre-requisite" for achieving agreement. Roughly speaking, if we say that a fact is a common knowledge amongst a group of agents then we mean that "Everybody knows that everybody knows that... everybody knows that ... everybody knows that fact". For further information about epistemic logic, see [FHMV2000].

In this work we present a tableaux method based on [CFGH1998] and it has the following features: the propagation of formulas are not necessarily top-down and the underlying structure is not necessarily a tree. Such approach and features can be seen in the work of [GM1996], [Massacci1994] and [CFGH1998]. Their approach also illustrates another feature also present here: single-step rules. These rules not only allow formulas to be added from one node to another (its successors or predecessor) in the tree (the structure) but also to sibyl nodes in the structure. The rules can be stated as "pattern-driven": applicable if a particular instance of a pattern in the structure is a pre-condition to trigger a rule application.

In [CFGH1998] was argued that "While trees are a good basis for many usual modal logics, they fail to support confluent relations for example. We argue in this paper that rooted directed acyclic graphs (RDAG for short), which are DAGs having a distinguished node called the root, are better suited. They allow to naturally handle some properties that do not fit easily with tree structures (like confluence, density), while other properties (like transitivity, symmetry,) can still be handled by propagation of formulas". We generalized their work to handle the epistemic modal logic by allowing more than one agents using multi rooted direct acyclic graphs ($MRDAG$'s [3]). The difficulties lay in how to propagate formulas through distinct relations (since we have more than one agent) and the absence of roots [4]. Another difficult is that several relations could have different relational properties (euclidianity, transitivity, density, confluence and so on). Finally the greatest challenge is to handle the common knowledge operator since it concerns different relation with all sorts of relational properties.

We can classify our tableaux rules into two sorts as far as relational properties are concerned:

1. Propagation rules: if there exists a pattern involving a formula that matches the precondition of a rule of this sort then create a new structure with the same components of previous structure and add a new formula (not necessary the one in the precondition) to the new structure. They correspond to axioms T (reflexivity), 5(euclidianity), B(symmetry) and 4(transitivity).
2. Structural rules: if there exists a pattern that matches the precondition of a rule of this sort then create a new structure with the same components of previous structure and add a new nodes or/and new edges to the new structure. They correspond to axioms D(seriality), De (density) and C(confluence).

We also can classified tableaux rules into three sorts as far as knowledge operators are concerned:

1. Classical rules: As well as the epistemic modal logic is a generalization of the classical modal logic (necessity/possibility), the classical modal logic is also a generalization of propositional logic. As result the epistemic modal logic still has propositional operators. This rule handle the propositional trait of this logic. They only add formulas to a specific node of the structure. Actually no knowledge operators are involved.
2. Epistemic rules: handle the epistemic operator K_i and its negation.
3. Common knowledge rules: handle the epistemic operator $\mathbf{C}_G$ and its negation. Where G is a set of agents [FHMV2000].

At the beginning we claimed that this tableaux method is a generalization of the one presented in [CFGH1998] for two reasons:

- "simplicity, naturality and modularity" are preserved: no restriction is made on the number of relations. The relations do not need to have the same set of relational properties. The classic, epistemic and common knowledge rules are completely independent as well as propagation and structural rules. The diversity of operators and their expressiveness are increased and enhanced with no tradeoffs. This tableaux method is also complete and, of course, sound.
- The classical modal logic can be regarded as logic that formalizes the epistemic state of a single agent. We only need to regard the *box* operator [5] equivalent to K_1 operator and only take into account Kripke models with exactly one relation (one agent).

Our tableaux calculi has the sub-formula property: only sub-formulas of the initial formula are propagated. Thus the usual argument of finiteness can be applied and provides a decidability result.

For further information on modal logic and epistemic modal logic: [Kripke1959], [Kripke1963], [Fitting1983], [Fitting1993], [Swart1980] and [FHMV2000].

[3] Shall be defined later

[4] A root is a node that can access every other node in transitive closure of a relation. Such feature was crucial in [CFGH1998] to their completeness and soundness results

[5] Box stands for "is necessary that"

2 Basic Definitions

Definition 1. *Let* R *be a binary relation.* $R(x) = \{y|(x,y) \in R\}$.

Definition 2. *Let* $M = (W, R_1, ..., R_n, \tau)$ *be a Kripke structure defined as follows:*

- W *is a non-empty set.*
- *For* $1 \leq i \leq n$, $R_i \subseteq W \times W$
- *Let* p *be an atom in a propositional language* $\wp$. p *holds in word* $w \in W$ iff $p \in \tau(w)$, *where* τ *is function over* W *that labels a world* $w \in W$ *to a subset of* $\wp$. *We also make the assumption that atoms are only the non-capitalized letters occurring in formulas.*

Definition 3. *Relational Properties Group 1: Let* R *be a binary relation.*

Property Name	*Condition*	*Symbol*	*Axiom*
Transitivity	$(x,y),(y,w) \in R \Rightarrow (x,w) \in R$	*Tr*	$4 = K_i p \rightarrow K_i K_i p$
Reflexivity	$(x,x) \in R$	*Ref*	$T = K_i p \rightarrow p$
Symmetry	$(x,y) \in R \Leftrightarrow (y,x) \in R$	*Sym*	$B = \neg K_i p \neg K_i p \rightarrow p$
Euclidianty	$(x,y),(x,w) \in R \Rightarrow (y,w) \in R$	*Eucl*	$5 = \neg K_i p \neg K_i p \rightarrow K_i p$

Definition 4. *Relational Properties Group 2: Let* R *be a binary relation.*

Property Name	*Condition*	*Symbol*	*Axiom*
Density	$R \subseteq R^2$	*Dens*	$De = \neg K_i \neg p \rightarrow \neg K_i K_i \neg p$
Seriality	$R^0 \subseteq (R \circ \overline{R})$	*Ser*	$D = K_i p \rightarrow \neg K_i \neg p$
Confluence	$(\overline{R} \circ R) \subseteq (R \circ \overline{R})$	*Conf*	$C = \neg K_i \neg K_i p \rightarrow K_i \neg K_i \neg p$

Definition 5. *Let* ρ^1 *and* ρ^2 *be sets of relational properties of group 1 and 2, respectively. Where* $\rho^1 \subseteq \{Tr, Sym, Eucl, Ref\}$, $\rho^2 \subseteq \{Conf, Ser, Dens\}$. *Further* ρ^1 *is always a maximal subset of properties of group 1. In other words, "including all those of group 1 which are consequence of it". E.g. KDeB4 is denoted by* $\{Sym, Tr, Eucl, Dens\}$ *since euclidianity is a consequence of it.* $\rho = \rho^1 \cup \rho^2$

Definition 6. *Let* ρ *be a set of relational properties of group 1 and group 2* iff $\rho = \rho^1 \cup \rho^2$

Definition 7. *Let* $\rho^1 = (\rho^1_1, ..., \rho^1_i, ..., \rho^1_n)$ *and* $\rho^2 = (\rho^2_1, ..., \rho^2_i, ..., \rho^2_n)$ *be n-tuples of sets of relational properties of group 1 and 2, respectively.* $\rho = \rho^1 \cup \rho^2 = (\rho^1_1 \cup \rho^2_1, ..., \rho^1_i \cup \rho^2_i, ..., \rho^1_n \cup \rho^2_n)$ *is a n-tuple of sets of relational properties.*

Definition 8. *Let* F, A *and* B *be epistemic modal formulas. Let* M *be a Kripke model and* w *a node in* M. *Let* $M, w \models F$ *be a model for a formula* F. $M, w \models F$ *holds* iff *it is one of the possibilities bellow:*

- $M, w \models p$ iff $p \in \tau(w)$ *and* p *is an atom.*
- $M, w \models \neg A$ iff *NOT* $M, w \models A$
- $M, w \models A \wedge B$ iff $M, w \models A$ *and* $M, w \models B$
- $M, w \models K_i A$ iff $\forall w'$: $(w, w') \in R_i$ *then* $M, w' \models A$ *(knowledge operator)*
- $M, w \models E_G A$ iff $\forall i \in G$: $M, w \models K_i A$. *Where* $G \subseteq \{1, ..., n\}$ *and* G *is non-empty.*
- $M, w \models E^m_G A$ iff *if* $m > 1$ *then* $M, w \models E^{m-1}_G E_G A$ *else* $M, w \models E_G A$.
- $M, w \models \boldsymbol{C}_G A$ iff $\forall i \in \{1, 2, 3, ...\}$: $M, w \models E^i_G A$. *(common knowledge operator)*

In [FHMV2000] we have an equivalent definition of common knowledge: "Our definition of common knowledge (above) has an interesting graph-theoretical interpretation, which turns out to be useful in many of our applications. Define a state t to be G-reachable from state s in k steps ($k \geq 1$) if there exist states s_0 , s_1, ..., s_k such that $s_0 = s$, $s_k = t$ and for all j with $0 \leq j \leq k-1$, there exists $i \in G$ such that $(s_j, s_{j+1}) \in R_i$. We say t is G-reachable from s if t is G-reachable from s in the k steps for some $k \geq 1$. Thus , t is G-reachable from s exactly if there is a path in the graph from s to t whose edges are labelled by members of G. In the particular case where G is the set of all agents, we say simply that t is reachable from s. Thus t is reachable from s and t are in the same connected component of the graph." Hence the following lemma will be quite useful in the correctness proof.

Lemma 1. $M, w \models \boldsymbol{C}_G A$ iff $M, t \models A$ *for all* t *that are* G*-reachable from* s*.*

Proof. See in *Lemma 2.2.1 b)* in [FHMV2000]

Lemma 2. $M, w \models \boldsymbol{C}_G A$ iff *if* $(w, t) \in (\bigcup_{i \in G} R_i)^+$ *then* $M, t \models A$*.*

Proof. Straightforward.

Definition 9. $(\aleph, \Sigma)$ *is an RGRAPH (Rooted Graph)* iff $\aleph$ *is non-empty set of nodes,* $\Sigma \subseteq \aleph \times \aleph$ *and there exists a special node called root in* $\aleph$ *that can access every node in the transitive closure of* Σ*.*

Definition 10. $(\aleph, \Sigma)$ *is an DAG (Direct Acyclic Graph)* iff $\aleph$ *is non-empty set of nodes and* $(\aleph, \Sigma)$ *is a graph such that* $\Sigma \subseteq \aleph \times \aleph$ *and* $(\aleph, \Sigma)$ *has no cycles.*

Definition 11. $(\aleph, \Sigma)$ *is an RDAG (Rooted Direct Acyclic Graph)* iff $(\aleph, \Sigma)$ *is a DAG and a RGRAPH.*

Definition 12. $(\aleph, \Sigma)$ *is an MRGRAPH (Multi Rooted Graph)* iff *there exists a sequence of* n *RGRAPH's,* $(\aleph_1, \Sigma_1)$*,... ,*$(\aleph_i, \Sigma_i)$*,... ,*$(\aleph_n, \Sigma_n)$ *such that* $\aleph = \bigcup_{i=1}^{n} \aleph_i$*,* $\Sigma = \bigcup_{i=1}^{n} \Sigma_i$ *and* $\forall i, j \in \{1, ..., n\}$*: if* $i \neq j$ *then* $\aleph_i \cap \aleph_j = \emptyset$*.*

Definition 13. $(\aleph, \Sigma)$ *is an MRDAG*[6] *(Multi Rooted Acyclic Graph)* iff *there exists a sequence of* n *RDAG's,* $(\aleph_1, \Sigma_1)$*,... ,*$(\aleph_i, \Sigma_i)$*,... ,*$(\aleph_n, \Sigma_n)$ *such that* $\aleph = \bigcup_{i=1}^{n} \aleph_i$*,* $\Sigma = \bigcup_{i=1}^{n} \Sigma_i$ *and* $\forall i, j \in \{1, ..., n\}$*: if* $i \neq j$ *then* $\aleph_i \cap \aleph_j = \emptyset$*.*

Let us define the non-empty set *rootset* of an *MRGRAPH*: a node n belongs to the *rootset* of an *MRGRAPH iff* n is the *root* of an *RGRAPH* contained in the *MRGRAPH*.

Definition 14 (ρ-Labelled Structure). *A* ρ*-labelled structure (*ρLS*) is a tuple* ρ-$(\aleph, \Sigma, FOR)$ *defined as follows:*

- $\aleph$*: a non-empty set nodes;*
- ρ*: is a n-tuple of sets of relational properties. Where* $\rho = (\rho_1, ..., \rho_i, ..., \rho_n)$*;*
- Σ*: is a n-tuple of sets of edges over* $\aleph$*. Where* $\Sigma = (\Sigma_1, ..., \Sigma_i, ..., \Sigma_n)$ *and* $\forall i, j \in \{1, ..., n\}$*: if* $i \neq j$ *then* $\Sigma_i \cap \Sigma_j = \emptyset$*;*
- ρ *and* Σ *have the same number of elements;*
- $\forall i :$ *if* $n \geq i \geq 1$ *then* $(\aleph, \Sigma_i)$ *is an MRDAG;*
- $(\aleph, \bigcup_{i=1}^{n} \Sigma_i)$ *is an RDAG;*
- *Let WWF be the set of all well formed formulas for the epistemic modal logic with the knowledge operators and the common knowledge operators. FOR is a finite binary relation such that* $FOR \subset \aleph \times WWF$*.* $FOR(x)$ *is the set of well formed formulas labelled to a world* x*. In other words, it is a function which maps a node to a set of formulas.*

[6] All *MRDAG* is also an *MRGRAPH*

For the sake of clarity, we will sometimes denote an *MRGRAPH* $(\aleph, \Sigma)$ only by its binary relation Σ.

Definition 15. *Let ρ be a n-tuple of sets of relational properties. A ρ-model is a Kripke model, $M = (W, R_1, ..., R_i, ..., R_n, \tau)$ such that each graph (W, R_i) satisfies every relational properties in ρ_i. A formula is ρ-satisfiable* iff *it is satisfiable in a ρ-model. It is ρ-valid (or a theorem)* iff *it is valid in the class of all ρ-models. This fact is denoted by $\models_\rho A$.*

3 Closure of *MRGRAPH*

Definition 16. *Let Σ be an MRGRAPH and ρ^1 be a set of relational properties of group 1. The ρ^1-closure of Σ (denotede by Σ^{ρ^1}) is the least MRGRAPH that contains Σ and which satisfies every property of ρ^1.*

Lemma 3 (General Structural Lema). *Let ρ^2 be a subset of group 2, ρ^1 be a subset of group 1 and let Σ be a ρ^2-MRGRAPH over a set of nodes $\aleph$. Then Σ^{ρ^1} is also a ρ^2-MRGRAPH and hence is a $(\rho^1 \cup \rho^2)$-MRGRAPH.*

Proof. Analogous to the Lemma 4.3 in [CFGH1998].

Lemma 4 (General Relational Closure Lemma). *Let Σ_i be an MRDAG over a set $\mathcal{N}$ of nodes.*

- $(x, y) \in \Sigma_i^{\rho_i=\{Ref\}}$ iff $(x, y) \in \Sigma_i$ *or* $x = y$
- $(x, y) \in \Sigma_i^{\rho_i=\{Sym\}}$ iff $(x, y) \in \Sigma_i$ *or* $(y, x) \in \Sigma_i$
- $(x, y) \in \Sigma_i^{\rho_i=\{Tr\}}$ iff $\exists n \geq 1\ (x, y) \in \Sigma_i^n$
- $(x, y) \in \Sigma_i^{\rho_i=\{Eucl\}}$ iff $(x, y) \in \Sigma_i$ *or* $\exists u \in \aleph\ \exists n \geq 1\ \exists m \geq 1$ *such that* $(u, x) \in \Sigma_i^n$ *and* $(u, x) \in \Sigma_i^m$
- $(x, y) \in \Sigma_i^{\rho_i=\{Ref,Sym\}}$ iff $(x, y) \in \Sigma_i$ *or* $(y, x) \in \Sigma_i$ *or* $x = y$
- $(x, y) \in \Sigma_i^{\rho_i=\{Ref,Tr\}}$ iff $\exists n \geq 1\ (x, y) \in \Sigma_i^n$ *or* $x = y$
- $(x, y) \in \Sigma_i^{\rho_i=\{Ref,Eucl\}}$ iff $\exists n \geq 0 \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = y : (x_i, x_{i+1}) \in \Sigma_i$ *or* $(x_{i+1}, x_i) \in \Sigma_i$
- $(x, y) \in \Sigma_i^{\rho_i=\{Sym,Tr\}}$ iff $\exists n \geq 1 \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = y : (x_i, x_{i+1}) \in \Sigma_i$ *or* $(x_{i+1}, x_i) \in \Sigma_i$
- $(x, y) \in \Sigma_i^{\rho_i=\{Tr,Eucl\}}$ iff $\exists u \in \aleph\ \exists n \geq 0\ \exists m \geq 1$ *such that* $(u, x) \in \Sigma_i^n$ *and* $(u, x) \in \Sigma_i^m$

Proof. It is a straightforward consequence of the General Closure under one Property Lemma and General Closure under Several Property Lemma. See the appendix.

Lemma 5. *Let Σ_i be an MRDAG over a set $\mathcal{N}$ of nodes.*

- $\Sigma_i^{\{Sym,Eucl\}} = \Sigma_i^{\{Sym,Tr,Eucl\}} = \Sigma_i^{\{Sym,Tr\}}$
- $\Sigma_i^{\{Ref,Sym,Tr\}} = \Sigma_i^{\{Ref,Tr,Eucl\}} = \Sigma_i^{\{Ref,Sym,Eucl\}} = \Sigma_i^{\{Ref,Sym,Eucl,Tr\}} = \Sigma_i^{\{Ref,Tr\}}$

Proof. Straightforward and analogous to the one in **Lemma 4.2** in [CFGH1998] and see also [FHMV2000].

Lemma 6 (Transitive Closure of the Union of Several Relations Under Several Properties Lemma). *Let Σ be n-tuple of MRGRAPH's over a set $\aleph$ and $G \subseteq \{1, ..., n\}$, where n is the cardinality of Σ and n is also the cardinality of the n-tuple ρ of group 1.*

$(x, y) \in (\bigcup_{i \in G} \Sigma_i^{\rho_i^1})^+$ iff $\exists h \geq 1,\ (w_0, w_1) \in \Sigma_{\alpha_1}^{\rho_{n1}^1 \cup \{Tr\}}, \ ...,\ (w_{j-1}, w_j) \in \Sigma_{\alpha_j}^{\rho_{\alpha_j}^1 \cup \{Tr\}}, \ ...,\ (w_{h-1}, w_h) \in \Sigma_{\alpha_h}^{\rho_{\alpha_h}^1 \cup \{Tr\}}$. *Where* $x = x_{w_0},\ y = x_{w_h},\ \forall j, 1 \leq j < h : \alpha_j \neq \alpha_{j+1}$ *and* $\alpha_i \in G$.

Proof. $(x,y) \in (\bigcup_{i\in G} \Sigma_i^{\rho_i})^+$

$\Leftrightarrow$
$\exists c \geq 1, (x_0, x_1) \in \Sigma_{a_1}^{\rho_{a_1}}, \ldots, (x_{i-1}, x_i) \in \Sigma_{a_i}^{\rho_{a_i}}, \ldots, (x_{c-1}, x_c) \in \Sigma_{a_c}^{\rho_{a_c}}$, where $x = x_0$, $y = x_c$ and $a_i \in G$.

$\Leftrightarrow$
$\exists h \geq 1,\ \exists b_1 \geq 1, x_{n_1}, ..., x_{n_1+(i-1)}, x_{n_1+i}, x_{n_1+i}, ..., x_{n_1+b_1} : (x_{n_1+(i-1)}, x_{n_1+i}) \in \Sigma_{a_{n_1}}^{\rho_{a_{n1}}}$

$\vdots$

$\exists b_j \geq 1, x_{n_j}, ..., x_{n_j+(i-1)}, x_{n_j+i}, x_{n_j+i}, ..., x_{n_j+b_j} : (x_{n_j+(i-1)}, x_{n_j+i}) \in \Sigma_{a_{n_j}}^{\rho_{a_{nj}}}$

$\vdots$

$\exists b_h \geq 1, x_{n_h}, ..., x_{n_h+(i-1)}, x_{n_h+i}, x_{n_h+i}, ..., x_{n_h+b_h} : (x_{n_h+(i-1)}, x_{n_h+i}) \in \Sigma_{a_{n_h}}^{\rho_{a_{nh}}}$

So that,
$a_{n1} = \ldots = a_{n_1+j} = \ldots = a_{n_1+b_1}$, where $n_1 = 1$ and $0 < j \leq b_1$

$\vdots$

$a_{n_i} = \ldots = a_{n_i+j} = \ldots = a_{n_i+b_i}$, where $n_i = n_{i-1} + b_{i-1} + 1$, $a_{n_i} \neq a_{n_{i-1}}$ and $0 < j \leq b_i$

$\vdots$

$a_{n_h+1} = \ldots = a_{n_h+j} = \ldots = a_{n_h+b_k}$, where $n_h = n_{h-1} + b_{h-1} + 1$, $a_{n_h} \neq a_{n_{h-1}}$ and $0 < j \leq b_{h-1}$. In this case, we have $a_c = a_{n_h+b_h}$

$x = x_{n_1}$, $y = x_{n_h+b_h}$, $\forall j, 1 \leq j < h : x_{n_{j+1}+b_j} = x_{n_{j+1}}$ and $a_i \in G$.

$\Leftrightarrow$
$\exists h \geq 1$, $(v_0, v_1) \in (\Sigma_{\alpha_1}^{\rho_{\alpha_1}})^{b_1}$, ..., $(v_{2(j-1)}, v_{2(j-1)+1}) \in (\Sigma_{\alpha_j}^{\rho_{\alpha_j}})^{b_j}$, ..., $(v_{2(h-1)}, v_{2(h-1)+1}) \in (\Sigma_{\alpha_h}^{\rho_{\alpha_h}})^{b_h}$

So that,
$\exists b_1 \geq 1, x_{n_1}, ..., x_{n_1+(i-1)}, x_{n_1+i}, ..., x_{n_1+b_1} : (x_{n_1}, x_{n_1+1}) \in \Sigma_{a_{n_1}}^{\rho_{a_{n1}}}$
$\exists b_j \geq 1, x_{n_j}, ..., x_{n_j+(i-1)}, x_{n_j+i}, ..., x_{n_j+b_j} : (x_{n_j+(i-1)}, x_{n_j+i}) \in \Sigma_{a_{n_j}}^{\rho_{a_{nj}}}$
$\exists b_h \geq 1, x_{n_h}, ..., x_{n_h+(i-1)}, x_{n_h+i}, ..., x_{n_h+b_h} : (x_{n_h+(i-1)}, x_{n_h+i}) \in \Sigma_{a_{n_h}}^{\rho_{a_{nh}}}$
$\forall j, 1 \leq j < h : v_{2(j-1)+1} = v_{2j}$, $(v_{2(j-1)}, v_{2(j-1)+1}) = (x_{n_j+b_j}, x_{n_{j+1}})$, $a_{n_j} \neq a_{n_{j+1}}$ and $a_{n_i} \in G$
$x = x_{n_1}$, $y = x_{n_h+b_h}$, $\forall j, 1 \leq j < h : x_{n_{j+1}+b_j} = x_{n_{j+1}}$, $\alpha_j \neq \alpha_{j+1}$, $\alpha_i = a_{n_i}$ and $\alpha_i \in G$

$\Leftrightarrow$
$\exists h \geq 1$, $(v_0, v_1) \in (\Sigma_{\alpha_1}^{\rho_{\alpha_1}})^+$, ..., $(v_{2(j-1)}, v_{2(j-1)+1}) \in (\Sigma_{\alpha_j}^{\rho_{\alpha_j}})^+$, ..., $(v_{2(h-1)}, v_{2(h-1)+1}) \in (\Sigma_{\alpha_h}^{\rho_{\alpha_h}})^+$

$x = x_{v_0}$, $y = x_{v_{2(h-1)+1}}$, $\forall j, 1 \leq j < h : v_{2(j-1)+1} = v_{2j}$, $\alpha_j \neq \alpha_{j+1}$ and $\alpha_i \in G$

$\Leftrightarrow$ (by **Transitivity Under Several Properties Lemma**. See the Appendix for further information)
$\exists h \geq 1$, $(v_0, v_1) \in \Sigma_{\alpha_1}^{\rho_{n_1} \cup \{Tr\}}$, ..., $(v_{2(j-1)}, v_{2(j-1)+1}) \in \Sigma_{\alpha_j}^{\rho_{\alpha_j} \cup \{Tr\}}$, ..., $(v_{2(h-1)}, v_{2(h-1)+1}) \in \Sigma_{\alpha_h}^{\rho_{\alpha_h} \cup \{Tr\}}$

$x = x_{v_0}$, $y = x_{v_{2(h-1)+1}}$, $\forall j, 1 \leq j < h : v_{2(j-1)+1} = v_{2j}$, $\alpha_j \neq \alpha_{j+1}$ and $\alpha_i \in G$
$\Leftrightarrow$

$\Leftrightarrow$
$\exists h \geq 1$, $(w_0, w_1) \in \Sigma_{\alpha_1}^{\rho_{n_1} \cup \{Tr\}}$, ..., $(w_{j-1}, w_j) \in \Sigma_{\alpha_j}^{\rho_{\alpha_j} \cup \{Tr\}}$, ..., $(w_{h-1}, w_h) \in \Sigma_{\alpha_h}^{\rho_{\alpha_h} \cup \{Tr\}}$
$x = x_{w_0}$, $y = x_{w_h}$, $\forall j, 1 \leq j < h : \alpha_j \neq \alpha_{j+1}$ and $\alpha_i \in G$

4 Pattern-Driven Rules

Let $\Psi = \rho\text{-}(\aleph, \Sigma, FOR)$ and $\bar{\Psi} = \bar{\rho}\text{-}(\bar{\aleph}, \bar{\Sigma}, F\bar{O}R)$ be ρLS's. The cardinality of Ψ and $\bar{\Psi}$ are c and $\bar{c}$, respectively. Later in this section we are going to present the rules grouped into three categories: Classic, Epistemic and Common Knowledge.

Definition 17. *Let Ψ and $\bar{\Psi}$ be ρLS's. $\bar{\Psi}$ is immediately derived from Ψ by r (alternatively, $\Psi \overrightarrow{r_i} \bar{\Psi}$ or $\bar{\Psi} \overleftarrow{r_i} \Psi$)* iff *r creates a new ρLS (for instance, $\bar{\Psi}$) such that $\aleph \subseteq \bar{\aleph}$, $c = \bar{c}$, $FOR \subseteq F\bar{O}R$ and $\forall i$ $1 \leq i \leq c : \Sigma_i \subseteq \bar{\Sigma}_i$ and $\rho_i = \bar{\rho}_i$.*

Definition 18. *Let Ψ and $\bar{\Psi}$ be ρLS's.$\bar{\Psi}$ is derived from Ψ (alternatively, $\Psi \Rightarrow \bar{\Psi}$ or $\bar{\Psi} \Leftarrow \Psi$)*iff *there are a sequence of ρLS's ($\bar{\Psi}^0$, ...,$\bar{\Psi}^i$, ...,$\bar{\Psi}^m$) and a sequence of rules (r_1, ..., r_i, ...r_m) where $1 \leq m$, $\Psi = \bar{\Psi}^0$, $\bar{\Psi} = \bar{\Psi}^m$ and $\forall i$ $1 \leq i < m$: $\bar{\Psi}^i \overleftarrow{r_i} \bar{\Psi}^{i-1}$. $\bar{\Psi}$ is also called a descendent of Ψ and Ψ is called an ancestor $\bar{\Psi}$.*

Definition 19. *Let r be a rule and Ψ and $\bar{\Psi}$ be ρLS's. r is applicable to Ψ* iff *r was not applied on a previous ancestor as far as its preconditions concerns and every preconditions of r holds. $\bar{\Psi}$ represents a ρLS resulting from such application.*

Now we present a list of "meta-rules"[7]. We can also name this rule schema as pattern-driven rule application method by the same reason.

Classic Rules

- `Rule` $\neg$: If $\neg\neg A \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, A)\}$.
- `Rule` $\bot$: If $\neg A$, $A \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, \bot)\}$.
- `Rule` $\wedge$: If $A \wedge B \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, A), (x, B)\}$.
- `Rule` $\vee$: If $\neg(A \wedge B) \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, C)\}$.
  Where $C = \neg A$ or $C = \neg B$.

Epistemic Rules

- `Rule` $\neg K$: If $\neg K_i A \in FOR(x)$ then $\bar{\aleph} = \aleph \cup \{y\}$, $F\bar{O}R = FOR \cup \{(y, \neg A)\}$ and
  $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma}_i = \Sigma_i \cup \{(x, y)\}$ else $\bar{\Sigma}_j = \Sigma_j$. Where c is the cardinality of Σ.
- `Rule` K: If $K_i A \in FOR(x)$ and $(x, y) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, A)\}$.

Epistemic Propagation Rules

- `Rule` T: If $K_i A \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, A)\}$.
- `Rule` 4: If $K_i A \in FOR(x)$ and $(x, y) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, K_i A)\}$.
- `Rule` B: If $K_i A \in FOR(x)$ and $(y, x) \in \Sigma_i$then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, A)\}$.

[7] Since in order to be be fully specified a rule need its preconditions fulfilled. In other words, a meta-rule with all its preconditions fulfilled results in rule

- `Rule 5.1`: If $K_iA \in FOR(x)$ and $(y,x) \in \Sigma_i$ and $(y,w) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(w, K_iA)\}$.
- `Rule 5.2`: If $K_iA \in FOR(x)$ and $(y,x) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, K_iA)\}$.
- `Rule 5.3`: If $K_iA \in FOR(x)$ and $(y,x) \in \Sigma_i$ and $(x,w) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(w, K_iA)\}$.

Common Knowledge Rules

- `Rule` $CK_{\neg K}$: If $\neg \mathbf{C}_G A \in FOR(x)$ then let i be such that $i \in G : \bar{\aleph} = \aleph \cup \{y\}$ and $F\bar{O}R = FOR \cup \{(x, C)\}$ and $\forall j, 1 \leq j \leq c$ if $j = i$ then $\bar{\Sigma_i} = \Sigma_i \cup \{(x,y)\}$ else $\bar{\Sigma_j} = \Sigma_j$. Where $C = \neg A$ or $C = \neg \mathbf{C}_G A$, and c is the cardinality of Σ.
- `Rule` CK_K: If $\mathbf{C}_G A \in FOR(x)$ and $(x,y) \in \Sigma_i$ and $i \in G$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, A)\}$.

Common Knowledge Propagation Rules

Let i be the index of a relation in G:

- `Rule` CK_T: If $\mathbf{C}_G A \in FOR(x)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(x, A)\}$.
- `Rule` CK_4: If $\mathbf{C}_G A \in FOR(x)$ and $(x,y) \in \Sigma_i$then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, \mathbf{C}_G A)\}$.
- `Rule` CK_B: If $\mathbf{C}_G A \in FOR(x)$ and $(y,x) \in \Sigma_i$then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, A), (y, \mathbf{C}_G A)\}$.
- `Rule` $CK_5.1$: If $\mathbf{C}_G A \in FOR(x)$ and $(y,x) \in \Sigma_i$ and $(y,w) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(w, \mathbf{C}_G A)\}$.
- `Rule` $CK_5.2$: If $\mathbf{C}_G A \in FOR(x)$ and $(y,x) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(y, \mathbf{C}_{\{i\}} A)\}$.
- `Rule` $CK_5.3$: If $\mathbf{C}_G A \in FOR(x)$ and $(y,x) \in \Sigma_i$ and $(x,w) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(w, \mathbf{C}_G A)\}$.

Structural Rules

Let i be the index of a relation in Σ:

- `Rule` D: If $x \in \aleph$ then $\bar{\aleph} = \aleph \cup \{y\}$, $F\bar{O}R = FOR$ and $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma_i} = \Sigma_i \cup \{(x,y)\}$ else $\bar{\Sigma_j} = \Sigma_j$. Where c is the cardinality of Σ.
- `Rule` $C0$: If $(x,y) \in \Sigma_i$ and $(x,w) \in \Sigma_i$ then $\bar{\aleph} = \aleph \cup \{z\}$, $F\bar{O}R = FOR$ and $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma_i} = \Sigma_i \cup \{(y,z),(w,z)\}$ else $\bar{\Sigma_j} = \Sigma_j$. Where c is the cardinality of Σ.
- `Rule` $C1$: If $(x,y) \in \Sigma_i$ then $\bar{\aleph} = \aleph \cup \{z\}$, $F\bar{O}R = FOR$ and $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma_i} = \Sigma_i \cup \{(y,z)\}$ else $\bar{\Sigma_j} = \Sigma_j$. Where c is the cardinality of Σ.
- `Rule` De: If $(x,y) \in \Sigma_i$ then $\bar{\aleph} = \aleph \cup \{z\}$, $F\bar{O}R = FOR$ and $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma_i} = \Sigma_i \cup \{(x,z),(z,y)\}$ else $\bar{\Sigma_j} = \Sigma_j$. Where c is the cardinality of Σ.

In order to define a tableaux method for a system denoted by $\rho = \rho^1 \cup \rho^2$ we must associate three set of rules (as far as ρ_i^1 e ρ_i^2 are concerned) to each Σ_i. They will be useful to select which rule can be applied. They are the following sorts:

- Obligatory: classical, epistemic and common knowledge rule sets. They must always be present in order to have a basic modal epistemic logic (axiom K).
- Group 1: epistemic propagation and common knowledge propagation rule sets.
- Group 2: structural rule set.

As far as epistemic propagation and structural rules are concerned we select the suitable rules according to the sets ρ_i^1 and ρ_i^2 by restricting their scope of action to Σ_i. By looking up the following table according to the content of ρ_i^1 and ρ_i^2 we select the epistemic propagation and structural rules that can be applied in the scope of Σ_i.

Classic	Epistemic	Common Knowledge	Structural	Relational Property	Group
$\neg$, $\perp$, $\wedge$, $\vee$	-	-	-	-	Obligatory
-	K	CK_K	-	-	Obligatory
-	$\neg K$	$CK_{\neg K}$		-	Obligatory
-	T	CK_T	-	Ref	1
-	4	CK_4	-	Tr	1
-	B	CK_B	-	Sym	1
-	5.1, 5.2, 5.3	$C_5.1$, $C_5.2$, $C_5.3$	-	Eucl	1
-	-	-	D	Ser	2
-	-	-	De	Dens	2
-	-	-	$C0$, $C1$	Conf	2

However in order to select the common knowledge propagation rule set for a particular relation Σ_i we must perform the following extra step: to create a η_i such that η_i is equal to the minimal superset that contains $\rho_i^1 \cup \{Tr\}$ and all properties that are consequence of such union. The purpose of this extra set is to add to the tableaux method the G-reach property inherent to the common knowledge. Why? Intuitively the "G-reach" trait is only transitivity not restricted to a single relation, but to the union of the relations belonging to G. It is not a coincidence that any instance of $\mathbf{C}_G A \to \mathbf{C}_G \mathbf{C}_G A$ is a valid formula whatever the class of models while $K_i A \to K_i K_i A$ is only valid in the class of models where R_i is transitive. By looking up the table above according to the content of η_i we select the common knowledge propagation rule that can be applied in the scope of Σ_i.

Definition 20. *A $(\rho^1 \cup \rho^2)$-tableaux for a formula A is the limit*[8] *of a sequence $\Psi_0, \ldots, \Psi_i, \Psi_{i+1}, \ldots$ where:*

- *Ψ_0 is an ρLS consisting of only one node whose associated set of formulas is $\{A\}$.*
- *Ψ_{i+1} is obtained from Ψ_i by applying either a obligatory rule, or a rule belonging to $(\rho^1 \cup \rho^2)$.*
- *and in which every applicable rule has been applied.*

Definition 21. *A tableaux is closed if* ***some*** *node in it contains $\perp$; Otherwise, it is open. A formula is $(\rho^1 \cup \rho^2)$-closed* iff ***all*** *its $(\rho^1 \cup \rho^2)$-tableaus are closed.*

5 Overview: Completeness and Soundness

In this section we only provide a "whole picture" of the completeness and soundness proofs. In the following sections we will prove them formally. The purpose of this section is to explain qualitatively the relation amongst the lemmas needed to accomplish the proof.

5.1 Completeness

In order to prove completeness we we build a ρ-model of a formula A from an open tableaux for a formula A. For that we needed to conceive and prove the following lemmas:

- **Knowledge Lemma**: states that whenever x and y are related in the closure of a particular relation, for instance R_i, and $K_i A \in FOR(x)$ then the set of associated rules assure that A is transported into y, if R_i such closure).

[8] Beware since there exists rules that generate multiples ρLS: $\vee$ and $CK_{\neg K}$ rules. Then we will have to take into account several sequences.

- **Common Knowledge Lemma**: states that whenever x and y are related in the transitive closure of the union of relations, for instance $(x, y) \in (\bigcup_{i \in G} R_i)^+$, and $\mathbf{C}_G \in FOR(x)$ $(i \in G)$ then the set of associated rules assures that A is transported into y only within the scope of each of R_i^+. Where $i \in G$.
- **Common Knowledge Negation Lemma**: states that if $\neg \mathbf{C}_G A \in FOR(x)$ in the structure then we can always derive a host of structures with another node y such that y is G-reachable [FHMV2000] from x and $\neg A$ is labelled to y. It is backed up by the **Transitive Closure of the Union of Several Relations under Several Properties Lemma**.
- **General Relational Closure Lemma**: states the properties of the closure of an $MRGRAPH$ under some relational properties in terms of an initial $MRDAG$ (Multi-Rooted Direct Acyclic Graph).
- **General Structural Lemma**: states that the closure of an $MRDAG$ under some relational properties preserves some of its initial features (e.g. the transitive closure of a confluent $MRDAG$ yields a confluent $MRGRPH$).
- **General Fundamental Lemma**: states that in the light of the lemmas above, the completeness proof goes smoothly by induction on the structure of a formula.

As we can see modularity is preserved.

5.2 Soundness

Straightforward: it is enough to use the embedding technique in [CFGH1998] and the "G-reach" property of common knowledge in [FHMV2000] by showing that the application of a rule preserve the satisfiability of the structure.

6 Completeness

Now we are going to prove that our tableaux method is complete. It is straightforward: from an open ρ-tableaux for a formula A we build a ρ-model of a formula A.

Definition 22. *Let $\Gamma = (W, R_1, ..., R_n, \tau)$ be the Krippe defined as bellow:*

- $W = \aleph$
- *For* $0 \leq i \leq n$, $R_i = \Sigma_i^{\rho_i^1}$
- *For all* $w \in W$, $p \in \tau(w)$ iff $p \in FOR(w)$, *where* p *is propositional symbol.*

As far as the relational proprieties are concerned, it is easy to see that the built model satisfies the properties of group 1 and according to the General Strutural Lemma it also satisfies the properties of group 2. As a result, it is a ρ-model. The remaining task it to show that such model satisfies *also* the formula A.

Lemma 7 (Knowledge Lemma).
Let $\Psi = (\rho^1 \cup \rho^2)$-$(\aleph, \Sigma, FOR)$ be a ρ-tableaux. Let $(x, y) \in \Sigma_j^{\rho_j^1}$, if $K_j A \in FOR(x)$ then $A \in FOR(y)$.

Proof. It is analogous to the **Lemma 7.1** in [CFGH1998]. There exist nine possible cases, as far as ρ_j^1 is concerned, where $j \in \{1, ..., c\}$ and c is the cardinality of Σ. We shall prove the lemma for the most awkward cases. All those cases involve euclidianity.

- $\rho_j^1 = \{Eucl\}$: if $(x,y) \in \Sigma_j^{\rho_j^1}$ then by General Reltional Closure Lemma, we have either $(x,y) \in \rho$ and then $A \in y$ and then $A \in y$ (by Rule K), or $\exists u \in \aleph\ \exists n \geq 1\ \exists m \geq 1$ such that $(u,x) \in \Sigma_j^n$ and $(u,y) \in \Sigma_j^m$. Therefore
  - $\exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = u : (x_{i+1}, x_i) \in \Sigma_j$; then $K_jA \in FOR(x_i)$ for $0 \leq i \leq n$(by Rule $S5.1$ n times), in particular: $K_jA \in FOR(x_n)$ and $x_n = u$
  - $\exists y_0 = u, ..., y_i, y_{i+1}, ..., y_m = y : (y_i, y_{i+1}) \in \Sigma_j$; So $K_jA \in FOR(y_1)$ (by rule $S5.1$ since $K_jA \in FOR(x_{n-1})$) from which we get $K_jA \in FOR(y_i)$, for $0 \leq i \leq m$ (by Rule $S5.3$ m times) and since $K_jA \in FOR(x_i)$ and $x_n = u = y_0$ it comes: $K_jA \in FOR(y_i)$for $0 \leq i \leq m$ (by Rule $S5.3$ m times). Then $A \in FOR(y_i)$ for $0 \leq i \leq m$ (by Rule K m times), in particular $A \in y$.
- $\rho_j^1 = \{Tr, Eucl\}$: if $(x,y) \in \Sigma_j^{\rho_j^1}$ then by General Reltional Closure Lemma, we have $\exists u \in \aleph$ $\exists n \geq 0\ \exists m \geq 1$ such that $(u,x) \in \Sigma_j^n$ and $(u,y) \in \Sigma_j^m$. Therefore
  - $\exists n \geq 0\ \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = u : (x_{i+1}, x_i) \in \Sigma_j$; then $K_jA \in FOR(x_0)$ implies for $K_jA \in FOR(u)$ (by Rule $S5.2$ n times)
  - $\exists m \geq 0\ \exists y_0 = y, ..., y_i, y_{i+1}, ..., y_{m+1} = y : (y_i, y_{i+1}) \in \Sigma_j$; then $K_jA \in FOR(u)$ implies for $K_jA \in FOR(y_m)$(by Rule 4 m times) and (by Rule K) $A \in y$.
- $\rho_j^1 = \{Sym, Tr, Eucl\}$: if $(x,y) \in \Sigma_j^{\rho_j^1}$ then by General Reltional Closure Lemma, we have $\exists n \geq 1\ \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = y : (x_i, x_{i+1}) \in \Sigma_j$ or $(x_{i+1}, x_i) \in \Sigma_j$; but $K_jA \in FOR(x_0)$ and $K_jA \in FOR(x_{i+1})$ due to $K_jA \in FOR(x_i)$ (by rule 4 or $S5.2$ according to whether $(x_i, x_{i+1}) \in \Sigma_j$ or $(x_{i+1}, x_i) \in \Sigma_j$). Thus $K_jA \in FOR(x_i)$ for $0 \leq i \leq n$ and hence $A \in FOR(x_i)$ for $0 \leq i \leq n+1$ (by rule K or B). Thus $A \in y$

Lemma 8 (Common Knowledge Lemma).
Let $\Psi = \rho$-$(\aleph, \Sigma, FOR)$ be a ρ-tableaux. Let c be the cardinality of Σ, $G \subseteq \{1, ..., c\}$, $(x,y) \in (\bigcup_{i \in G} \Sigma_j^{\rho_i^1})^+$, if $\mathbf{C}_G A \in FOR(x)$ then $A \in FOR(y)$.

Proof. According to Transitive Closure of the Union of Several Relations under Several Properties Lemma, the rules that handle the common knowledge operator need only to act locally. In other words they must propagate the formulas in the scope of specific relations indexed by the G set of a common knowledge operator (e.g. $\mathbf{C}_G$). Since two or more relations may share nodes then the patterns or sub-patterns of the form $\mathbf{C}_G A$ will be propagated so that global effect (the transitive closure of the union of relation) of the common knowledge operator will be captured and reproduced by the common knowledge rules. The same lemma states that given a relation Σ_j and a set ρ_j^1 then it will suffice to move that patterns around the $\Sigma_j^{\eta_j}$. Where $\eta_j = \rho_j^1 \cup \{Tr\}$, $j \in G \subseteq \{1, ..., c\}$ and c is the cardinality of Σ. The following proof is analogous to the **Lemma 7.1** in [CFGH1998]. There exist five possible cases, as far as η_j is concerned. We shall prove the lemma for the most awkward cases. All those cases involve euclidianity.

- $\eta_j = \{Tr, Eucl\}$: if $(x,y) \in \Sigma_j^{\eta_j}$ then by General Reltional Closure Lemma, we have $\exists u \in \aleph$ $\exists n \geq 0\ \exists m \geq 1$ such that $(u,x) \in \Sigma_j^n$ and $(u,y) \in \Sigma_j^m$. Therefore
  - $\exists n \geq 0\ \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = u : (x_{i+1}, x_i) \in \Sigma_j$; then $\mathbf{C}_G A \in FOR(x_0)$ implies for $\mathbf{C}_{\{j\}} A \in FOR(u)$ (by Rule $CK_5.2$ n times).
  - if $n > 0$ then $G' = \{j\}$ else $G' = G$. $\exists m \geq 0\ \exists y_0 = y, ..., y_i, y_{i+1}, ..., y_{m+1} = y : (y_i, y_{i+1}) \in \Sigma_j$; then $\mathbf{C}_{G'} A \in FOR(u)$ implies for $\mathbf{C}_{G'} A \in FOR(y_m)$(by Rule CK_4 m times) and (by Rule CK_K) $A \in y$.
- $\eta_j = \{Sym, Tr, Eucl\}$: if $(x,y) \in \Sigma_j^{\eta_j}$ then by General Reltional Closure Lemma, we have $\exists n \geq 1\ \exists x_0 = x, ..., x_i, x_{i+1}, ..., x_n = y : (x_i, x_{i+1}) \in \Sigma_j$ or $(x_{i+1}, x_i) \in \Sigma_j$; but $\mathbf{C}_G A \in FOR(x_0)$ and $\mathbf{C}_{G_{i+1}} A \in FOR(x_{i+1})$ due to $\mathbf{C}_{G_i} A \in FOR(x_i)$ where $G_{i+1} = G_i$ or $G_{i+1} = \{j\}$ (by rule

CK_4 or $CK_5.2$ according to whether $(x_i, x_{i+1}) \in \Sigma_j$ or $(x_{i+1}, x_i) \in \Sigma_j$, respectively). Thus $\mathbf{C}_{G_i} A \in FOR(x_i)$ for $0 \leq i \leq n$ and hence $A \in FOR(x_i)$ for $0 \leq i \leq n+1$ (by rule CK_K or CK_B). Thus $A \in y$.

Lemma 9 (Auxiliar Lemma).

Let $a_1, ..., a_i, ..., a_n$ be a sequence where $a_i \in G$, Ψ a tableaux. If $\neg \mathbf{C}_G A \in FOR(x)$ then there exists two tableaus, Ψ_n and $\bar{\Psi}_n$ by appliance of the rule $CK_{\neg K}$ n times where $\Psi_n \Leftarrow \Psi$ and $\bar{\Psi}_n \Leftarrow \Psi$,$\forall i$ $1 \leq i < n$ $[(x_{2i-1}, x_{2i}) \in \Sigma_{a_i}, (x_{2i-1}, x_{2i}) \in \bar{\Sigma}_{a_1}, x_{2i} = x_{2i+1}]$, $(x_{2n-1}, x_{2n}) \in \Sigma_{a_n}, (x_{2n-1}, x_{2n}) \in \bar{\Sigma}_{a_n}$, $\neg A \in FOR_n(x_{2n})$ and $\neg \mathbf{C}_G A \in F\bar{O}R_n(x_{2n})$.

Proof. By induction on the length of $a_1, ..., a_i, ..., a_n$.

Induction Initialization: suppose that the rule $CK_{\neg K}$ was not applied to $a_i \in G$, Ψ. So we have two possible outcomes (by Rule $CK_{\neg K}$): $\bar{\bar{\Psi}}_1 \overleftarrow{CK_{\neg K}} \Psi$ and $\bar{\Psi}_1 \overleftarrow{CK_{\neg K}} \Psi$. In particular, let the new edges be $(x, y) \in \bar{\bar{\Sigma}}_{a_1}$ and $(x, y) \in \bar{\bar{\Sigma}}_{a_1}$, in each tableaux. Since we have two choices (Rule $CK_{\neg K}$) we have $F\bar{\bar{O}}R_1(y) = \{\neg A\}$ and $F\bar{O}R_1(y) = \{\neg \mathbf{C}_G A\}$. We make $\Psi_1 = \bar{\bar{\Psi}}_1$. We make $(x, y) = (x_1, x_2)$. Then the lemma holds for $n = 1$.

Induction Step: Let $n = m$. Then (by Inductive Hypothesis) $\bar{\Psi}_m$ the following statements hold: $\forall i$ $1 \leq i < m$ $[(x_{2i-1}, x_{2i}) \in \bar{\Sigma}_{a_1}, x_{2i} = x_{2i+1}]$, $(x_{2n-1}, x_{2m}) \in \bar{\Sigma}_{a_m}$, $\neg \mathbf{C}_G A \in F\bar{O}R_m(x_{2m})$ for $\bar{\Psi}_m$. So we have two possible outcomes (by Rule $CK_{\neg K}$): $\bar{\bar{\Psi}}_{m+1} \overleftarrow{CK_{\neg K}} \bar{\Psi}_m$ and $\bar{\Psi}_{m+1} \overleftarrow{CK_{\neg K}} \bar{\Psi}_m$. In particular, let the new edges be $(x_{2m}, y) \in \bar{\Sigma}_{a_{m+1}}$ and $(x_{2m}, y) \in \bar{\bar{\Sigma}}_{a_{m+1}}$ where $a_{m+1} \in G$. Since we have two choices (Rule $CK_{\neg K}$) we have $F\bar{\bar{O}}R_{m+1}(y) = \{\neg A\}$ and $F\bar{O}R_m(y) = \{\neg \mathbf{C}_G A\}$. We make $\Psi_{a_{m+1}} = \bar{\bar{\Psi}}_{a_{m+1}}$. We also make $(x_{2m}, y) = (x_{2m+1}, x_{2(m+1)})$. Then the lemma holds for $n = m+1$.

Lemma 10 (Common Knowledge Negation Lemma). *Let $\Psi = \rho$-$(\aleph, \Sigma, FOR)$ be a ρ-tableaux. Let c be the cardinality of Σ, $G \subseteq \{1, ..., c\}$, if $\neg \mathbf{C}_G A \in FOR(x)$ and every possible rule has be applied then $\exists y$: $(x, y) \in (\bigcup_{i \in G} \Sigma_i^{\rho_i^1})^+$ $\neg A \in FOR(y)$.*

Proof. Suppose $\neg \mathbf{C}_G A \in FOR(x)$. if there is no $(x, y) \in \Sigma_{a \in G}$ such that $\neg A \in FOR(y)$ then Rule $CK_{\neg K}$ can be applied n times by choosing any sequence $(a_1, ..., a_i, ..., a_n)$. Where $a_i \in G$. (Otherwise this lemma holds trivially since $\Sigma_{a \in G} \subseteq \Sigma_{a \in G}^{\rho_i^1}$). According to **Auxiliary Lemma** there exists derivation that

$\exists n \geq 1$, $(x_1, x_2) \in \Sigma_{a_1}$, ..., $(x_{2j-1}, x_{2j}) \in \Sigma_{a_j}$, ..., $(x_{2n}, x_{2n+1}) \in \Sigma_{a_n}$, $\forall j, 1 \leq j < h$: $x_{n_{j+1}+b_j} = x_{n_{j+1}}$, $a_i = a_{n_i}$, $a_i \in G$, $x = x_1$ and $y = x_{2n}$

$\Leftrightarrow$

$\exists n \geq 1$, $(w_0, w_1) \in \Sigma_{a_1}$, ..., $(w_j, w_{j+1}) \in \Sigma_{a_j}$, ..., $(w_{n-1}, w_n) \in \Sigma_{a_n}$, $a_i \in G$, $x = w_1$ and $y = w_n$

$\Leftrightarrow$

$\exists n \geq 1$, $(x, y) \in (\bigcup_{a_j \in G} \Sigma_{a_j})^n$

$\Leftrightarrow$

$(x, y) \in (\bigcup_{a_j \in G} \Sigma_{a_j})^+ \subseteq (\bigcup_{a_j \in G} \Sigma_{a_j}^{\rho_j^1})^+$ (by growth and idempotence).

Since we have make no special claim, this lemma holds $\forall x \in \mathcal{N}$: $\neg \mathbf{C}_G A \in FOR(x)$. Otherwise there would be an applicable rule.

Lemma 11 (General Fundamental Lemma). *Let $\Psi = \rho$-$(\aleph, \Sigma, FOR)$ be an open ρ-tableaux for a formula A. Let Γ be the model built from Ψ as above (definition 22). Let $B \in Subformulas(A)$. We have that (I) if $B \in FOR(x)$ then $\Gamma, x \models B$*

Proof. Induction on the size of B.

Induction Initialization: Let B be an atom; then (a) holds by definition of τ.

Induction Step:

- B cannot be $\bot$, else x would be closed.
- Let B be $\neg\neg C$.
  $\neg\neg C \in x \Rightarrow C \in x$ (by Rule $\neg$) $\Rightarrow \Gamma, x \models C$ (by II) $\Rightarrow \Gamma, x \models \neg\neg C$
- Let B be $C \wedge D$.
  $C \wedge D \in x \Rightarrow C \in x$ and $D \in x$ (by Rule $\wedge$) $\Rightarrow \Gamma, x \models C$ and $\Gamma, x \models D$ (by II) $\Rightarrow \Gamma, x \models C \wedge D$
- Let B be $\neg(C \wedge D)$.
  $C \wedge D \in x \Rightarrow \neg C \in x$ or $\neg D \in x$ (by Rule $\vee$) $\Rightarrow \Gamma, x \models \neg C$ or $\Gamma, x \models \neg D$ (by II) $\Rightarrow$ $\Gamma, x \models \neg(C \wedge D)$
- Let B be $\neg\mathbf{K}_i C$.
  $\neg\mathbf{K}_i C \in x$
  $\Rightarrow$ Exists y such that $(x, y) \in \Sigma_i$ and $\neg C \in y$ (by Rule $\neg\mathbf{K}$)
  $\Rightarrow$ Exists y such that $(x, y) \in R_i$ and $\Gamma, y \models \neg C$ (by II and since $\Sigma_i \subseteq R_i$)
  $\Rightarrow$ Exists y such that $(x, y) \in R_i$ and $\neg C \in y$ (by Rule $\neg\mathbf{K}$)
  $\Rightarrow \Gamma, x \models \neg\mathbf{K}_i C$
- Let B be $\mathbf{K}_i C$.
  Suppose $(x, y) \in R_i$; then by the **Knowledge Lemma**, $C \in y$. So by (II), we have $\Gamma, y \models C$. As a result, $\Gamma, x \models \mathbf{K}_i C$
- Let B be $\mathbf{C}_G C$.
  Suppose $(x, y) \in (\bigcup_{i \in G} R_i)^+$; then by the **Common Knowledge Lemma**, $C \in y$. So by (II), we have $\Gamma, y \models C$. As a result, $\Gamma, x \models \mathbf{C}_G C$
- Let B be $\neg\mathbf{C}_G C$.
  By the **Common Knowledge Negation Lemma**, $\exists y$: $(x, y) \in (\bigcup_{i \in G} \Sigma_i^{\rho_i^1})^+$ and $\neg A \in FOR(y)$. So by (II), we have $\Gamma, y \models \neg C$. As a result, $\Gamma, x \models \neg\mathbf{C}_G C$

Corollary 1. *If a formula A has open ρ-tableaux then A is ρ-satisfiable. Hence our tableaux method is complete.*

7 Soundness

Now we will prove that our tableaux method is sound. if a formula A is $(\rho^1 \cup \rho^2)$-closed then A is $(\rho^1 \cup \rho^2)$-unsatisfiable [CFGH1998]. We will show that all rules are "satisfiability" preserving as far as the resulting ρLS from the application of one of them is concerned. In our sense, a pattern is $(\rho^1 \cup \rho^2)$-satisfiable *iff* there exists $(\rho^1 \cup \rho^2)$-model that contains it and satisfies its formulas [CFGH1998].

Definition 23. *Let $\Psi = (\rho^1 \cup \rho^2)$-$(\aleph, \Sigma, FOR)$ be a labelled $(\rho^1 \cup \rho^2)$-$MRGRAPH$ and $\mu = (W, R, \tau)$ be a $(\rho^1 \cup \rho^2)$-model. Where $R = \{R_1, ..., R_c\}$ and c is the cardinality of Σ. Let h be a function such that $h(\aleph) \subseteq W$ and $\forall i \in \{1, ..., c\}$, $\forall n_1, n_2 \in \aleph : (n_1, n_2) \in \Sigma_i \Rightarrow (h(n_1), h(n_2)) \in R_i$.*

- *h is called an embedding from Ψ to μ (or h matches Ψ to μ),*
- *μ satisfies Ψ via h* iff *$\forall n \in \aleph : A \in FOR(n) \Rightarrow$ to $\mu, h(n) \models A$,*
- *μ satisfies Ψ* iff *there exists an embedding h from Ψ to μ such that μ satisfies Ψ via h.*

Lemma 12. *Let r be a rule of some property belonging to group 1 or 2 and Ψ, $\overline{\Psi}_1$,...,$\overline{\Psi}_i$,...,$\overline{\Psi}_m$ be tableaux such that $\Psi \vec{r} \overline{\Psi}_1$ or ... or $\Psi \vec{r} \overline{\Psi}_i$ or ... or $\Psi \vec{r} \overline{\Psi}_m$. If some ρ-model μ satisfies Ψ then it satisfies $\overline{\Psi}_1$ or ... or $\overline{\Psi}_i$ or ... or $\overline{\Psi}_m$. Where $0 < i \leq m$ (depending on r).*

Proof. Initially we suppose the existence of a model μ such that μ satisfies Ψ via some embedding h. Thereby we display an embedding h' that μ satisfies $\overline{\Psi}$ via h'. It is accomplished by analyzing every rule and its possible outcomes. For classical rules, it is immediate: just take $h' = h$. We prove soundness for a couple of classical rules, obligatory rules, the most awkward cases (involving euclidianity) and an example of structural rule:

- $\Psi \overrightarrow{\nabla} \overline{\Psi}_1$ or $\Psi \overrightarrow{\nabla} \overline{\Psi}_2$ $(m = 2)$: $\neg(A \wedge B) \in FOR(n_0)$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(n_0, C)\}$. Where $C = \{(n_0, \neg A)\}$ or $C = \{(x, \neg B)\}$. if μ satisfies Ψ via h then $\mu, h(n_0) \models \neg(A \wedge B)$. Then we have that $\mu, h(n_0) \models \neg A$ or $\mu, h(n_0) \models \neg B$ hold.
- $\Psi \overrightarrow{\neg K} \overline{\Psi}_1$ $(m = 1)$: $\neg K_i A \in FOR(n_0)$ then $\bar{\aleph} = \aleph \cup \{n_1\}$, $\bar{\Sigma}_i = \Sigma_i \cup \{(n_0, n_1)\}$ and $F\bar{O}R = FOR \cup \{(n_1, \neg A)\}$. if μ satisfies Ψ via h then $\mu, h(n_0) \models \neg K_i A$, hence $\exists y \in R_i(h(n_0)) : \mu, y \models \neg A$; let y_1 be such a y, and define $h'(n_1) = y_1$ and $\forall j \in \aleph : h'(j) = h(j)$. μ satisfies $\overline{\Psi}$ via h', since $(h'(n_0), h'(n_1)) \in R_i$ and $\mu, h'(n_1) \models A$.
- $\Psi \overrightarrow{CK_{\neg K}} \overline{\Psi}_1$ or, ..., or $\Psi \overrightarrow{CK_{\neg K}} \overline{\Psi}_{2c}$ (c is the cardinality of G):$\neg \mathbf{C}_G A \in FOR(n_0)$ then let i be such that $i \in G : \bar{\aleph} = \aleph \cup \{n_1\}$ and $F\bar{O}R = FOR \cup \{(n_1, C)\}$ and $\forall j, 1 \leq j \leq c$ if $j = i$ then $\Sigma'_i = \Sigma_i \cup \{(n_0, n_1)\}$ else $\Sigma'_j = \Sigma_j$. Where $C = \neg A$ or $C = \neg \mathbf{C}_G A$, and c is the cardinality of Σ. Since μ satisfies Ψ via h then $\mu, h(n_0) \models \neg \mathbf{C}_G A$. Hence $\exists y : y$ is G-reachable from n_0 and $\mu, y \models \neg A$ and let y' be such y and let such path from $h(n_0)$ to y' be denoted by $\exists b > 0 : h(n_0) = x_0, x_1 ..., x_j, x_{j+1}, ..., x_b = y'$ such that $(x_j, x_{j+1}) \in R_{r_j}$ where $r_j \in G$ *iff*
  1. $(x_0, x_1) \in R_{r_0}$ where $r_0 \in G$, $x_0 = h(n_0)$, $x_1 = y$ and $b = 1$, or
  2. $\exists b > 1 : h(n_0) = x_0, x_1 ..., x_j, x_{j+1}, ..., x_b = y'$ such that $(x_j, x_{j+1}) \in R_{r_j}$ where $r_j \in G$, $(x_0, x_1) \in R_{r_0}$.

  If $r_0 = i$ then $x_1 \in R_i(h(n_0))$ and $\mu, x_1 \models C$; Now we define $h'(n_1) = x_1$ and $\forall j \in \aleph : h'(j) = h(j)$. Therefore μ satisfies $\overline{\Psi}$ via h', such that $(h'(n_0), h'(n_1)) \in R_i$, $\mu, h'(n_1) \models C$ and (since $x_b = y$) if $b = 1$ then $C = \neg A$ (as y G-reachable from x_0) otherwise $C = \neg \mathbf{C}_G A$ (as y G-reachable from x_1 and x_1 G-reachable from x_0).
- $\Psi \overrightarrow{De} \overline{\Psi}_1$: If $(n_0, n_1) \in \Sigma_i$ then $\bar{\aleph} = \aleph \cup \{n_2\}$, $F\bar{O}R = FOR$ and $\forall j, 1 \leq j \leq c$: if $j = i$ then $\bar{\Sigma}_i = \Sigma_i \cup \{(n_0, n_2), (n_2, n_1)\}$ else $\bar{\Sigma}_j = \Sigma_j$. Where c is the cardinality of Σ. If μ satisfies Ψ via h then $(h(n_0), h(n_1)) \in R_i$, and since R is dense $\exists z : (h(n_0), z) \in R_i$ and $(z, h(n_1)) \in R_i$. Let z_2 be such a z and define $h'(n_2) = z_2$ and $h'(n) = h(n)$ for $n \neq n_2$. μ satisfies $\overline{\Psi}$ via h', since $(h'(n_0), h'(n_2)) \in R_i$, $(h'(n_2), h'(n_1)) \in R_i$ and $F\bar{O}R(n_2) = \emptyset$

For propagation rules, we just have to prove that we re done by taking $h' = h$.

- $\Psi \overrightarrow{S5.1} \overline{\Psi}_1$ $(m = 1)$: $K_i A \in FOR(n_1)$ and $(n_0, n_1) \in \Sigma_i$ and $(n_0, n_2) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(n_2, K_i A)\}$. Suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models K_i A$ but $\mu, h(n_2) \models K_i A$ does not hold. Then we have that $\mu, h(n_2) \models \neg K_i A$ holds. Hence $\exists y \in R_i(h(n_2)) : \mu, y \models \neg A$. Let y' be such y. Since R_i is euclidian and $(h(n_0), h(n_1))$, $(h(n_0), h(n_2))$, $(h(n_2), y') \in R_i$ then $(h(n_1), y) \in R_i$. According to the semantic definition of the K_i operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(h(n_2)) \models K_i A$ holds.
- $\Psi \overrightarrow{S5.2} \overline{\Psi}_1$ $(m = 1)$: $K_i A \in FOR(n_1)$ and $(n_0, n_1) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(n_0, K_i A)\}$. Suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models K_i A$ but $\mu, h(n_0) \models K_i A$ does not hold. Then we have that $\mu, h(n_0) \models \neg K_i A$ holds. Hence $\exists y \in R_i(h(n_0)) : \mu, y \models \neg A$. Let y' be such y. Since R_i is euclidian and $(h(n_0), h(n_1))$, $(h(n_0), y') \in R_i$ then $(h(n_1), y) \in R_i$. According to the semantic definition of the K_i operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(n_0) \models K_i A$ holds.
- $\Psi \overrightarrow{S5.3} \overline{\Psi}_1$ $(m = 1)$: $K_i A \in FOR(x)$ and $(n_0, n_1) \in \Sigma_i$ and $(n_1, n_2) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $F\bar{O}R = FOR \cup \{(n_2, K_i A)\}$. Suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models K_i A$

but $\mu, h(n_2) \models K_i A$ does not hold. Then we have that $\mu, h(n_2) \models \neg K_i A$ holds. Hence $\exists y \in R_i(h(n_2)) : \mu, y \models \neg A$. Let y' be such y. Since R_i is euclidian and $(h(n_0), h(n_1))$,$(h(n_1), h(n_2))$, $(h(n_2), y') \in R_i$ then $(h(n_1), y) \in R_i$. According to the semantic definition of the K_i operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(n_0) \models K_i A$ holds.

- $\Psi \overrightarrow{CK_5.1} \Psi_1$ or, ..., or $\Psi \overrightarrow{CK_5.1} \Psi_c$ (c is the cardinality of G): $\mathbf{C}_G A \in FOR(n_1)$ and $(n_0, n_1) \in \Sigma_i$ and $(n_0, n_2) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $\bar{FOR} = FOR \cup \{(n_2, \mathbf{C}_G A)\}$. Let $i \in G$ and suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models \mathbf{C}_G A$ but $\mu, h(n_2) \models \mathbf{C}_G A$ does not hold. Then we have that $\mu, h(n_2) \models \neg \mathbf{C}_G A$ holds. Hence $\exists y : y$ is G-reachable from $h(n_2)$ and $\mu, y \models \neg A$. Let y' be such y. Since R_i is euclidian and $(h(n_0), h(n_1))$, $(h(n_0), h(n_2)) \in R_i$ then $(h(n_1), h(n_2)) \in R_i$. As a result y' is G-reachable from $h(n_1)$. According to the semantic definition of the $\mathbf{C}_G$ operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(n_2) \models \mathbf{C}_G A$ holds.
- $\Psi \overrightarrow{CK_5.2} \Psi_1$ or, ..., or $\Psi \overrightarrow{CK_5.2} \Psi_c$ (c is the cardinality of G): $\mathbf{C}_G A \in FOR(n_1)$ and $(n_0, n_1) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $\bar{FOR} = FOR \cup \{(n_0, \mathbf{C}_{\{i\}} A)\}$. Let $i \in G$ and suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models \mathbf{C}_G A$ but $\mu, h(n_0) \models \mathbf{C}_{\{i\}} A$ does not hold. Then we have that $\mu, h(n_0) \models \neg \mathbf{C}_{\{i\}} A$ holds. Hence $\exists y : y$ is $\{i\}$-reachable from n_0 and $\mu, y \models \neg A$ and let y' be such y and let such path from n_0 to y' be denoted by $\exists h > 0 : h(n_0) = x_0, x_1 ..., x_j, x_{j+1}, ..., x_h = y'$ such that $(x_j, x_{j+1}) \in R_i$. Since R_i is euclidian and $(h(n_0), h(n_1))$, $(h(n_0), x_1) \in R_i$ then $(h(n_1), x_1) \in R_i$. As a result y' is G-reachable from n_1. According to the semantic definition of the $\mathbf{C}_G$ operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(n_0) \models \mathbf{C}_{\{i\}} A$ holds.
- $\Psi \overrightarrow{CK_5.3} \Psi_1$ or, ..., or $\Psi \overrightarrow{CK_5.3} \Psi_c$ (c is the cardinality of G): $\mathbf{C}_G A \in FOR(n_1)$ and $(n_0, n_1) \in \Sigma_i$ and $(n_1, n_2) \in \Sigma_i$ then $\bar{\aleph} = \aleph$, $\bar{\Sigma} = \Sigma$ and $\bar{FOR} = FOR \cup \{(n_2, \mathbf{C}_G A)\}$. Suppose that μ satisfies Ψ via h then $\mu, h(n_1) \models \mathbf{C}_G A$ but $\mu, h(n_2) \models \mathbf{C}_G A$ does not hold. Then we have that $\mu, h(n_2) \models \neg \mathbf{C}_G A$ holds. Hence $\exists y : y$ is G-reachable from n_2 and $\mu, y \models \neg A$. Let y' be such y. Since R_i is euclidian and $(h(n_0), h(n_1))$,$(h(n_1), h(n_2)) \in R_i$ then As a result y' is G-reachable from $h(n_1)$. According to the semantic definition of the $\mathbf{C}_G$ operator we have that $\mu, y' \models A$. What is a contradiction. Then $\mu, h(n_2) \models \mathbf{C}_G A$ holds.

8 Decidability and Termination

Our tableaux calculi has the sub-formula property: only sub-formulas of the initial formula are propagated. Thus the usual argument of finiteness can be applied and provides a decidability result. Further in [FHMV2000] (chapter 3) we found a discussion about the decidability of the validity problem for epsitemic modal logic involving common knowledge. See also [Fitting1983] about blocking the creation of repeated nodes in an $RGRAPH$.

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10 Appendix

Definition 24. *Binary relation classes:*

- *The set of all binary relation will be denoted as* $\mathcal{R}$
  $\mathcal{R}$ *is set of all relation over a given set. Let* $R, S \in \mathcal{R}$*:*
  - $(x, y) \in R^n$ iff *exists a path in* R *of length* n *from* x *to* y*.*
  - $(x, y) \in \overline{R}$ iff $(x, y) \in R$.
  - $(x, y) \in I$ iff $x = y$ *and* $x \in \mathcal{N}$*.* I *is called diagonal relation. It also is represented as* R^0
  - $(x, y) \in R \circ S$ iff *exists* $w : (x, w) \in R$ *and* $(w, y) \in S$
  - $(x, y) \in \mathcal{U}$ iff $x \in \mathcal{N}$ *and* $y \in \mathcal{N}$
  - $R(x)$ iff $\{y \in \mathcal{N} : (x, y) \in R\}$.
- *The set of all rooted relation will be denoted as* $\mathcal{RR}$
- *The set of all multi-rooted relation will be denoted as* $\mathcal{MRR}$ *[CFGH1998].*
  Let $G = \{1, ..., n\}$*,* R *a relation over* $\mathcal{N}$ *and* $R \in \mathcal{MRR}$ iff *for all* $i \in G$ *and for all* $j \in G$*:*
  - R_i *is a relation over* $\mathcal{N}_i$ *and* $R_i \in \mathcal{RR}$
  - $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$
  - $R = \bigcup_{i=1}^{n} R_i$
  - $\mathcal{N} = \bigcup_{i=1}^{n} \mathcal{N}_i$

  The Immediate consequences of the definition of $\mathcal{MRR}$ are

- if $i \neq j$ then $R_i \circ R_j = \emptyset$, for R_i and R_j satisfying the four conditions above, since $\mathcal{N}_i \neq \mathcal{N}_j$.
- $\mathcal{R} \supset \mathcal{MRR} \supset \mathcal{RR}$

For the sake of clarity, from now on if $R, S \in \mathcal{MRR}$ than they have the same index set G such that: $\forall i \in G$: both R_i and S_i are relations over N_i

Property 1: Let $R \in \mathcal{R}$ [CFGH1998]

1. $R^+ = \bigcup_{i \geq 1} R^i$, transitive closure
2. $\overline{\overline{R}} = R$
3. $\overline{R \cup S} = \overline{R} \cup \overline{S}$
4. $\overline{R \circ S} = \overline{R} \circ \overline{S}$
5. $\overline{R^n} = \overline{R}^n$, for $n \geq 0$
6. $\overline{R^+} = \overline{R}^+$
7. $\overline{R^*} = \overline{R}^*$
8. $(R \cup I)^+ = R^*$, transitive and reflexive closure
9. $(R \cup S) \circ T = (R \circ T) \cup (S \circ T)$
10. $T \circ (R \cup S) = (T \circ R) \cup (T \circ S)$
11. $I = I^+ = I^* = \overline{I}$
12. if $R \neq \emptyset$ then $R \circ \overline{R} \neq \emptyset$
13. if $R \neq \emptyset$ then $\mathcal{U} \circ \mathcal{R} \circ \mathcal{U} = \mathcal{U}$
14. $(R^n)^+ \subseteq (R^+)^n$, for $n \geq 0$
15. $R \subseteq R^\rho$, growth
16. $R \subseteq S \Rightarrow R^\rho \subseteq S^\rho$, monotonicity
17. if $P \in \rho$ then $(R^\rho)^P \subset R^\rho$, idempotence; and of course: $(R^\rho)^\rho = R^\rho$
18. R is reflexive *iff* $I \subseteq R$
19. R is symmetrical *iff* $\overline{R} \subseteq R$
20. R is transitive *iff* $R^2 \subseteq R$
21. R is euclidean *iff* $(\overline{R} \circ R) \subseteq R$, or *iff* $(\overline{R} \circ R) \subseteq \overline{R}$

22. R is dense *iff* $R \subseteq R^2$
23. R is serial *iff* $I \subseteq (R \circ \overline{R})$
24. R is confluent *iff* $(\overline{R} \circ R) \subseteq (R \circ \overline{R})$
25. R is rooted *iff* $(\overline{R^*} \circ R^*) = \mathcal{U}$
26. R is connected *iff* $(\overline{R} \cup R)^* = \mathcal{U}$, rooted implies connected.

Property 2: Let R $\in \mathcal{RR}$ [CFGH1998]

1. Let ρ be a subset of group 1 then $R^\rho \in \mathcal{RR}$
2. $\overline{R^+} \circ R^+ \circ \overline{R^+} \circ R^+ = \overline{R^+} \circ R^+$
3. if $\overline{R} \circ R$ then $\overline{R^+} \circ R^+$
4. $(\overline{R^*} \circ R^+)^+ \subseteq \overline{R^*} \circ R^+$

Property 3: Let R and $S \in \mathcal{MRR}$

1. $R \circ R = \bigcup_{i=1}^n R_i \circ R_i$, R and $R \circ R \in \mathcal{MRR}$
2. $R^m = \bigcup_{i=1}^n R_i^m$, R and $R^m \in \mathcal{MRR}$
3. $\overline{R}$ and $R \in \mathcal{MRR}$
4. $R^+ = \bigcup_{i=1}^n R_i^+$, R and $R^+ \in \mathcal{MRR}$
5. $R^* = \bigcup_{i=1}^n R_*^*$, R and $R^* \in \mathcal{MRR}$
6. $R \circ S = \bigcup_{i=1}^n R_i \circ S_i$, R,S and $R \circ S \in \mathcal{MRR}$
7. Let ρ be a subset of group 1 then $R^\rho \in \mathcal{RR}$
8. $\overline{R^+} \circ R^+ \circ \overline{R^+} \circ R^+ = \overline{R^+} \circ R^+$
9. if $\overline{R} \circ R$ then $\overline{R^+} \circ R^+$
10. $(\overline{R^*} \circ R^+)^+ \subseteq \overline{R^*} \circ R^+$

Proof. 1. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ R_j = \emptyset$
2. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ R_j = \emptyset$
3. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ R_j = \emptyset$
4. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ S_j = \emptyset$
5. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ S_j = \emptyset$
6. Trivial since $i \neq j \Leftrightarrow \mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ then $R_i \circ S_j = \emptyset$
7. Trivial since the *root set* of R is still a root set in R^ρ
8. $\overline{R^+} \circ R^+ \circ \overline{R^+} \circ R^+ = \overline{\bigcup_{i=1}^n R_i^+} \circ \bigcup_{i=1}^n R_i^+ \circ \overline{\bigcup_{i=1}^n R_i^+} \circ \bigcup_{i=1}^n R_i^+ = \bigcup_{i=1}^n \overline{R_i^+} \circ \bigcup_{i=1}^n R_i^+ \circ \bigcup_{i=1}^n \overline{R_i^+} \circ \bigcup_{i=1}^n R_i^+ = \bigcup_{i=1}^n \overline{R_i^+} \circ R_i^+ \circ \overline{R_i^+} \circ R_i^+ = \bigcup_{i=1}^n \overline{R_i^+} \circ R_i^+ = \bigcup_{i=1}^n \overline{R_i^+} \circ \bigcup_{i=1}^n R_i^+ = \overline{R^+} \circ R^+$
9. $\overline{R} \circ R = \bigcup_{i=1}^n \overline{R}_i \circ \bigcup_{i=1}^n R_i = \bigcup_{i=1}^n \overline{R}_i \circ R_i \Rightarrow \bigcup_{i=1}^n \overline{R}_i^+ \circ R_i^+ = \bigcup_{i=1}^n \overline{R}_i^+ \circ \bigcup_{i=1}^n R_i^+ = \overline{R}^+ \circ R^+$
10. $(\overline{R^*} \circ R^+)^+ = (\bigcup_{i=1}^n \overline{R_i^*} \circ \bigcup_{i=1}^n R_i^+)^+ = (\bigcup_{i=1}^n \overline{R_i^*} \circ R_i^+)^+ = \bigcup_{i=1}^n (\overline{R_i^*} \circ R_i^+)^+ \subseteq \bigcup_{i=1}^n \overline{R_i^*} \circ R_i^+ = \bigcup_{i=1}^n \overline{R_i^*} \circ \bigcup_{i=1}^n R_i^+ = \overline{R^*} \circ R^+$

Now we are going to present the most import lemma needed to prove completeness.

Lemma 13 (Closure under one Property Lemma).
Where $R \in \mathcal{RR}$:

1. $R^{Ref} = R \cup I$
2. $R^{Sym} = R \cup \overline{R}$
3. $R^{Tr} = R^+$
4. $R^{Eucl} = R \cup (\overline{R^+} \circ R^+)$

Proof. See [CFGH1998].

Lemma 14 (General Closure under one Property Lemma). *Where* $R \in \mathcal{MRR}$

1. $R^{Ref} = R \cup I$
2. $R^{Sym} = R \cup \overline{R}$
3. $R^{Tr} = R^{+}$
4. $R^{Eucl} = R \cup (\overline{R^{+}} \circ R^{+})$

Proof. The items 1,2,3 are obvious and well-known since they hold for general relations. We will prove only the item 4. We will show that that relation (i) $R \cup (R^{+} \circ \overline{R^{+}})$ is euclidian. Then we will prove that (ii) $R \cup (R^{+} \circ \overline{R^{+}}) \subseteq R^{Eucl}$. Then we will come to the conclusion since R^{Eucl} is the least superset of R being euclidian and, as such, it contains any other euclidian superset of R.

(i) $(R \cup (\overline{R^{+}} \circ R^{+})) \circ \overline{(R \cup (\overline{R^{+}} \circ R^{+}))} =$
$= (\bigcup_{i=1}^{n} R_i \cup (\overline{(\bigcup_{i=1}^{n} R_i)^{+}} \circ (\bigcup_{i=1}^{n} R_i)^{+})) \circ \overline{(\bigcup_{i=1}^{n} R_i \cup (\overline{(\bigcup_{i=1}^{n} R_i)^{+}} \circ (\bigcup_{i=1}^{n} R_i)^{+}))} =$
$= (\bigcup_{i=1}^{n} R_i \cup (\bigcup_{i=1}^{n} \overline{R_i^{+}} \circ \bigcup_{i=1}^{n} R_i^{+})) \circ \overline{(\bigcup_{i=1}^{n} R_i \cup (\bigcup_{i=1}^{n} \overline{R_i^{+}} \circ \bigcup_{i=1}^{n} R_i^{+}))} =$
$= \bigcup_{i=1}^{n} (R_i \cup (\overline{R_i^{+}} \circ R_i^{+})) \circ \overline{\bigcup_{i=1}^{n} (R_i \cup (\overline{R_i^{+}} \circ R_i^{+}))} =$
$= \bigcup_{i=1}^{n} (R_i \cup (\overline{R_i^{+}} \circ R_i^{+})) \circ \overline{(R_i \cup (\overline{R_i^{+}} \circ R_i^{+}))}$
Since $R_i \in \mathcal{RR} \Rightarrow \bigcup_{i=1}^{n} (R_i \cup (\overline{R_i^{+}} \circ R_i^{+})) \circ \overline{(R_i \cup (\overline{R_i^{+}} \circ R_i^{+}))} \subseteq \bigcup_{i=1}^{n} R_i = R$

Proof. (ii) The proof presented here is analogue to the one presented in [CFGH1998]. First we prove by induction on $i+j$ that $\forall i, j : (\overline{R^i} \circ R^j) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$.

Induction Initialization:
$i+j=2$, i.e. $i=j=1$: $(\overline{R} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$. Since $R \subset R^{Eucl}$ and $\overline{R} \subset \overline{R^{Eucl}}$.

Induction Step:
if $j > 1$
then $(\overline{R^i} \circ R^j) = (\overline{R^i} \circ R^{j-1} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl} \circ R)$ (by II) $\subseteq (\overline{R^{Eucl}} \circ R) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$.
else if $j = 1$ and $i > 1$
then $(\overline{R^i} \circ R^j) = (\overline{R} \circ \overline{R^{i-1}} \circ R^j) \subseteq (\overline{R} \circ \overline{R^{Eucl}} \circ R^{Eucl})$ (by II) $\subseteq (R \circ \overline{R^{Eucl}}) \subseteq (\overline{R^{Eucl}} \circ R^{Eucl})$.

Now, since $(\overline{R^{+}} \circ R^{+}) = \overline{\bigcup_{i=1} R^i} \circ \bigcup_{j \geq 1} R^j = \bigcup_{i \geq 1, j \geq 1} \overline{R^i} \circ R^j \subseteq \bigcup_{i \geq 1, j \geq 1} \overline{R^{Eucl}} \circ R^{Eucl} = \overline{R^{Eucl}} \circ R^{Eucl} \subseteq R^{Eucl}$. So we obtain $R \cup (\overline{R^{+}} \circ R^{+}) \subset R \cup R^{Eucl} \subseteq R^{Eucl}$

Lemma 15 (Auxiliary Property Lemma). *Let* $\rho = \{P_1, ..., P_n\}$ *be a subset of group 1, and* $R \in \mathcal{MRR}$. *Let* $R_0 = R$ *and* $R_{i+1} = (...(R_i^{P_1})...)^{P_n}$*; then if there exists* m *that* $R_{m+1} = R_m$ *then* $R_m = R^{\rho}$.

Proof. Straightforward and analogous to the one in **Lemma 9.2** in [CFGH1998].

Lemma 16 (General Closure under Several Property Lemma). *Where* $i \in G$ *and* $\Sigma_i \in \mathcal{MRR}$

1. $R^{Ref,Sym} = R \cup \overline{R^{+}} \cup I$
2. $R^{Ref,Tr} = (R \cup I)^{+}$
3. $R^{Ref,Sym,Tr} = (R \cup \overline{R} \cup I)^{+}$
4. $R^{Sym,Tr} = (R \cup \overline{R})^{+}$

5. $R^{Tr,Eucl} = \overline{R^{*}} \circ R^{+}$

Proof. We indicate a closure by some property x by $\overrightarrow{X}$:

1. $R \ \overrightarrow{REF} \ R \cup I \ \overrightarrow{SYM} \ R \cup I \cup \overline{R \cup I} = R \cup \overline{R} \cup I \ \overrightarrow{SYM} \ R \cup \overline{R} \cup I$
2. $R \ \overrightarrow{REF} \ R \cup I \ \overrightarrow{TR} \ (R \cup I)^{+} \ \overrightarrow{REF} \ (R \cup I)^{+} \cup I = (R \cup I)^{+} = (R \cup I)^{*}$
3. $R \ \overrightarrow{REF} \ R \cup I \ \overrightarrow{SYM} \ R \cup I \cup \overline{R \cup I} = R \cup \overline{R} \cup I \ \overrightarrow{SYM} \ R \cup \overline{R} \cup I \ \overrightarrow{TR} \ (R \cup \overline{R} \cup I)^{+} \ \overrightarrow{REF} \ (R \cup \overline{R} \cup I)^{+} \cup I = (R \cup \overline{R} \cup I)^{+} = (R \cup \overline{R})^{*} = \bigcup_{n \geq i} (R_i \cup \overline{R_i})^{*} = \bigcup_{n \geq i} \mathcal{U}_i \ \overrightarrow{SYM} \ \bigcup_{n \geq i} \mathcal{U}_i \cup \overline{\mathcal{U}}_i = \bigcup_{n \geq i} \mathcal{U}_i = (R \cup \overline{R})^{*}$
4. $R \ \overrightarrow{SYM} \ R \cup \overline{R} \ \overrightarrow{TR} \ (R \cup \overline{R})^{+} \ \overrightarrow{SYM} \ (R \cup \overline{R})^{+} \cup [\overline{(R \cup \overline{R})}]^{+} = (R \cup \overline{R})^{+} \cup \overline{(R \cup \overline{R})^{+}} = (R \cup \overline{R})^{+} \cup (\overline{R} \cup \overline{\overline{R}})^{+} = (R \cup \overline{R})^{+}$
5. $R \ \overrightarrow{TR} \ R^{+} \ \overrightarrow{EUCL} \ R^{+} \cup (\overline{(R^{+})^{+}} \circ (R^{+})^{+}) = R^{+} \cup (\overline{R^{+}} \circ R^{+}) = (I \circ R^{+}) \cup (\overline{R^{+}} \circ R^{+}) = (I \cup \overline{R^{+}}) \circ R^{+} = \overline{R^{*}} \circ R^{+} \ \overrightarrow{TR} = (\overline{R^{*}} \circ R^{+})^{+} = (\overline{R^{*}} \circ R^{+})$ (About $\mathcal{RR}$)

Lemma 17 (Transitivity under Several Properties Lemma).

1. $(R^{Ref})^{+} = R^{Ref,Tr}$
2. $(R^{Sym})^{+} = R^{Sym,Tr}$
3. $(R^{Tr})^{+} = R^{Tr}$
4. $(R^{Eucl})^{+} = R^{Eucl,Tr}$
5. $(R^{Ref,Sym})^{+} = R^{Ref,Sym,Tr}$
6. $(R^{Ref,Tr})^{+} = R^{Ref,Tr}$
7. $(R^{Ref,Sym,Tr})^{+} = R^{Ref,Sym,Tr}$
8. $(R^{Sym,Tr})^{+} = R^{Sym,Tr}$
9. $(R^{Tr,Eucl})^{+} = R^{Tr,Eucl}$

Proof. By cases:

1. $(R^{Ref})^{+} = (R \cup I)^{+} = R^{*}$
2. $(R^{Sym})^{+} = (R \cup \overline{R})^{+}$
3. $(R^{Tr})^{+} = (R^{+})^{+} = R^{+}$
4. $(R^{Eucl})^{+} = (\overline{R^{*}} \circ R^{+})^{+} = \overline{R^{*}} \circ R^{+}$
5. $(R^{Ref,Sym})^{+} = (R \cup \overline{R} \cup I)^{+}$
6. $(R^{Ref,Tr})^{+} = ((R \cup I)^{+})^{+} = (R \cup I)^{+}$
7. $(R^{Ref,Sym,Tr})^{+} = ((R \cup \overline{R} \cup I)^{+})^{+} = (R \cup \overline{R} \cup I)^{+}$
8. $(R^{Sym,Tr})^{+} = ((R \cup \overline{R})^{+})^{+} = (R \cup \overline{R})^{+}$
9. $(R^{Tr,Eucl})^{+} = (\overline{R^{*}} \circ R^{+})^{+} = (\overline{R^{*}} \circ R^{+})$