PERTURBED PROJECTION METHOD FOR GENERAL VARIATIONAL INEQUALITIES

PAULO DOS SANTOS AND SUSANA SCHEIMBERG

Abstract. We consider the problem of General nonmonotone Variational Inequalities, (GVI), in a finite dimensional space. We propose a method to solve (GVI) that at each iteration it is considered only one projection on an easy approximation of the constraint set which is important from a practical point of view. We analyse the convergence of the algorithm under the assumption that the solution set of (GVI) is nonempty and a weak cocoercivity condition, using variational metric analysis. Preliminary computational experience is reported. A comparative analysis with two algorithms is also given for the monotone case.

1. Introduction

Let C be a nonempty closed convex subset of $\mathbb{R}^n$ and let F, g be operators from $\mathbb{R}^n$ into itself. We consider the following General Variational Inequality problem:

$$(1.1) \qquad (GVI): \begin{cases} \text{Find } \ x^* \in \mathbb{R}^n, \quad g(x^*) \in C : \\ \langle F(x^*), g(x) - g(x^*) \rangle \geq 0 \quad \forall \ \ g(x) \ \in C, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ be the Euclidean inner product.

The basic requirements for this problem are that F and g are continuous functions and $C \subset rge(g) = \{y \in \mathbb{R}^n : y = g(x) \text{ for some } x \in \mathbb{R}^n\}$.

This formulation was introduced by M. A. Noor [14] in 1988. It has applications in many fields including engineering, economics and operations research, see [4],[6], [8], [15], [16], [24] and the references therein. In [16] is proved that the odd-order obstacle problem is properly included in the above formulation. When the operator g is the identity, GVI becomes the classical Variational Inequality problem [11]. The generalized nonlinear complementarity problem and optimization programs can also be embedded in this setting, see for example [14], [21]. Existence conditions for problem GVI are also analysed in [21]. The algorithms proposed for solving GVI can be split into two classes: explicit ([14], [17], [23]) and implicit algorithms ([12], [17], [18], [19], [20]). The projection technique and its variants are extensively considered by both types of methods. When the constraint set does not have a special structure the projection can be hard to be solved. Actually, the numerical experiments given in the literature to solve GVI only consider box or ball constraint set (see for example [12], [19]). The convergence of explicit methods is guaranteed under strong monotonicity and Lipschitz continuity conditions on the applications F and g. The g-monotonicity of the operator F and the existence of the inverse function g^{-1} are common hypotheses required by implicit methods.

In this work we develop an implicit projection method, called PPM, for solving GVI that deals with easy projections related to approximations of the constraint set C, which is important from a practical point of view. We observe that at the moment we don't know on other method to solve GVI working with approximated sets. A second advantage

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of the method PPM is that only one projection and one inversion of operator g per iteration are required, reducing computational costs. In addition, we don't require the usual assumptions on F and g, that is, F is not necessarily g-monotone and the inverse function of g may not be defined. In the convergence analysis, like other authors, we assume that the solution set of GVI is nonempty. Under the assumptions that F is g-cocoercive related only to one solution of GVI we obtain a subsequence of the g-iterates, $g(x^k)$ or of the iterates x^k converging to a solution, depending if F is also g-Lipschitz on C or if the point-to-set application g^{-1} is . In both cases it is numerically easy to be identify the convergent subsequence. When we assume the existence of the inverse function g^{-1}, we obtain stronger results. We consider examples to show the difference and weightiness of each convergence result. We illustrate the application of our method with a numerical example using approximations of the constraint set C. We present a preliminary numerical performance of our algorithm by comparing it with the projection methods given in [12] and [19] for the monotone case.

The paper is organized as follows. In section 2, we recall useful basic notions. In section 3 we define the algorithm and we analyse its convergence. In section 4, we report a preliminary numerical experience. Finally, we have a conclusion section.

2. Basic preliminaries

2.1. **Introduction.** In this section we present notions that will be useful in next sections. We summarize some definitions and properties of projection operators. We also recall results of the variational analysis related to epiconvergence of sets and variational metric of operators. We conclude this part with a property connecting these notions.

2.2. **D-projection.** Let D be a symmetric positive definite matrix of order n. Given a nonempty closed convex subset C of $I\!R^n$, the D-projection operator on C related to the norm induced by D (D-norm), is defined by

$$P_C^D : I\!R^n \to I\!R^n, \quad P_C^D(x) = \arg\min_{y \in C} \left\{ \frac{1}{2} \|x - y\|_D^2 \right\} \tag{2.1}$$

where $\|v\|_D^2 =< v, v >_D =< v, Dv >$ for all $v \in I\!R^n$.

Remark 2.1. Given $x, y, z \in I\!R^n$ and $\lambda \in [0, 1]$, the following equalities hold for D-norms:

$$\text{i)} \quad \|\lambda x + (1-\lambda) y\|_D^2 = \lambda \|x\|_D^2 + (1-\lambda)\|y\|_D^2 - \lambda(1-\lambda)\|x - y\|_D^2 \tag{2.2}$$

$$\text{ii)} \quad \|x - z\|_D^2 - \|x - y\|_D^2 = -\|z - y\|_D^2 - 2\langle x - z, D(z - y)\rangle. \tag{2.3}$$

The next two results are the characterization of projections and the nonexpansivity property of the projection operator (Zarantonello [25]) applied to the D-projection operator.

Lemma 2.1. *Let $x \in I\!R^n$. The point p_x is the projection $P_C^D(x)$ if, and only if, it holds that*

$$\langle p_x - x, D(p_x - y)\rangle \le 0 \quad \forall \; y \in C. \tag{2.4}$$

Lemma 2.2. *The D-projection operator on C is nonexpansive related to the D-norm,*

$$\|P_C^D(x) - P_C^D(y)\|_D \le \|x - y\|_D \quad \forall \; x, y \in I\!R^n.$$

2.3. **Variational analysis.** Let us denote by $NCCS(\mathbb{R}^n)$ the family of nonempty closed convex subsets of $\mathbb{R}^n$. We begin this subsection by defining the γ_D-distance on $NCCS(\mathbb{R}^n)$ based in the notions given in [3] and [22] for functions and for maximal monotone operators.

Definition 2.1. Let $\gamma \geq 0$. The γ_D-distance on $NCCS(\mathbb{R}^n)$ is given by

$$d_\gamma^D(C_1, C_2) := \sup_{\|x\|_D \leq \gamma} \|P_{C_1}^D(x) - P_{C_2}^D(x)\|_D \tag{2.5}$$

for all $C_i \in NCCS(\mathbb{R}^n),\ i = 1, 2.$

The γ_D-distance corresponds to the function distance given in [3] applied to indicator functions of sets. The definition (2.5) also can be considered as the operator distance defined in [22] when the operators are subdifferentials of indicators functions. The next property is concerning to a well known characterization of the epiconvergence of sets (see for example Proposition 3.21, [2]).

Proposition 2.3. *Let C and C_k be sets in $NCCS(\mathbb{R}^n)$, for $k \in \mathbb{N}$. The following assertions are equivalents :*
a) The sequence $\{C_k\}$ epiconverges to C $\quad (C_k \xrightarrow{epi} C)$;
b) $\begin{cases} i)\ \forall\ x\ \in C,\ \exists\ \{x^k\}\ \mid\ x^k \in C_k\ \ \forall\ \ k \in \mathbb{N},\ \ and\ x^k \longrightarrow x; \\ ii)\ If\ \ \forall\, k \in \overline{N} \subset \mathbb{N}\colon x^k \in C_k,\ and\ \ x^k \longrightarrow x,\ then,\ \ x \in C; \end{cases}$

The following result is a direct consequence of the definition of epiconvergence and Corollary 2.53 given in [3].

Lemma 2.4. *Let C and C_k be in $NCCS(\mathbb{R}^n)$, for $k \in \mathbb{N}$. The following assertions are equivalents :*
a) $d_\gamma^D(C_k, C) \longrightarrow 0$, for all $\gamma \geq 0$;
b) $C_k \xrightarrow{epi} C$.

We conclude this section with a convergence result relating D-projections, epiconvergence of sets and convergence of points, that will be used in the development of our convergence analysis.

Proposition 2.5. *Let C and C_k be in $NCCS(\mathbb{R}^n)$, for $k \in \mathbb{N}$. If the sequence $\{C_k\}$ is epiconvergent to C and the sequence $\{x^k\} \subset \mathbb{R}^n$ is convergent to x, then, it holds*

$$\lim_{k \to +\infty} P_{C_k}^D(x^k) = P_C^D(x).$$

Proof. First, we prove that the accumulation points of the sequence $\{P_{C_k}^D(x^k)\}$ is nonempty by showing that it is bounded. In fact, given $P_C^D(x) \in C$, we have that there exists a sequence $\{y^k \in C_k\}$ converging to $P_C^D(x)$. Using that $y^k = P_{C_k}^D(y^k)$ for all $k \in N$, we obtain that

$$\begin{aligned} \|P_{C_k}^D(x^k) - P_C^D(x)\|_D &\leq \|P_{C_k}^D(x^k) - y^k\|_D + \|y^k - P_C^D(x)\|_D \\ &= \|P_{C_k}^D(x^k) - P_{C_k}^D(y^k)\|_D + \|y^k - P_C^D(x)\|_D. \end{aligned}$$

Since $\{x^k\}$ and $\{y^k\}$ are convergent sequences, by applying Lemma 2.2 it follows that $\{P_{C_k}^D(x^k) - P_{C_k}^D(y^k)\}$ is bounded, hence $\{P_{C_k}^D(x^k) - P_{C_k}^D(x)\}$ and $\{P_{C_k}^D(x^k)\}$ also are bounded. Now, to complete the proof it is sufficient to show that the sequence $\{P_{C_k}^D(x^k)\}$ has a unique accumulation point given by $P_C^D(x)$. Indeed, let $\{P_{C_j}^D(x^j)\}_{j \in \overline{N}}$ be a subsequence converging to $\overline{x}$. Thus, the characterization of epiconvergence it must be $\overline{x} \in C$.

Now, we show that $\overline{x} = P_C^D(x)$ by proving that $\langle \overline{x} - x, D(\overline{x} - w)\rangle \leq 0$ for all $w \in C$. Indeed, consider an arbitrary $w \in C$. Since $\{C_k\}$ epiconverges to C, it follows that there is a sequence $\{w^k \in C_k\}$ converging to w. By the D-projection characterization it follows that

$$\langle P_{C_j}^D(x^j) - x^j, D(P_{C_j}^D(x^j) - w^j)\rangle \leq 0 \ \forall j \in \overline{N}.$$

Passing onto the limit we get that

$$\langle \overline{x} - x, D(\overline{x} - w)\rangle \leq 0,$$

that is, $\overline{x} = P_C^D(x)$. Hence, $P_C^D(x)$ is the unique accumulation point of $\{P_{C_k}^D(x^k)\}$. This completes the proof. □

3. Perturbed Projection Method

3.1. **Introduction.** In order to present the algorithm PPM, we specify requirements on the approximation sets $\{C_k\}$ of C and on the parameters $\{\lambda_k\}$ used in the algorithm. We also indicate the notation we use.

R1 . $\{C_k\} \subset NCCS(I\!R^n)$ and it holds $C_k \subset C_{k+1} \subset C \subset rge(g), \ \ k \in I\!N$; $C_k \xrightarrow{epi} C$.

R2 . $\{\lambda_k\} \subset (0, 1]$.

Let $D \in I\!R^{n\times n}$ be a symmetric positive definite matrix and let α be a positive parameter, we consider the following notations:

$T(x) := P_C^D[g(x) - \alpha D^{-1}F(x)]$;

$T_k(x) := P_{C_k}^D[g(x) - \alpha D^{-1}F(x)]$;

$R_k(x) := g(x) - T_k(x)$;

$e^k := T_k(x^k) - T(x^k)$.

Remark 3.1. Let $x \in I\!R^n$ and $\gamma \geq \|g(x) - \alpha D^{-1}F(x)\|_D$. For each $k \in I\!N$ it holds that

$$\begin{aligned} \|T_k(x) - T(x)\|_D &\leq \|P_{C_k}^D[g(x) - \alpha D^{-1}F(x)] - P_C^D[g(x) - \alpha D^{-1}F(x)]\| \\ &\leq d_\gamma^D(C_k, C). \end{aligned} \tag{3.1}$$

Remark 3.2. A point x is a solution of problem GVI if, and only if, x is a g-fixed point of T, that is, $g(x) = T(x)$. Indeed, it is a direct consequence of Lemma 2.1. and the fact that $C \subset rge(g)$. This characterization is a natural extension of the well known result given by Eaves [7], when the operator g is the identity.

Taking into account the last remark, we define the following algorithm to solve GVI, by finding a g-fixed point of T.

3.2. **Algorithm PPM.**

Initialization : Choose $x^0 \in I\!R^n$ such that $g(x^0) \in C_0$, set $k = 0$.

Iterative step : Let $x^k \in I\!R^n$ such that $g(x^k) \in C_k$.

If $\|R_k(x^k)\|_D = 0$ then,

Null step: let $x^{k+1} := x^k$

Otherwise,

Serious step: find x^{k+1} such that

$$g(x^{k+1}) = g(x^k) - \lambda_k R_k(x^k) = (1 - \lambda_k)g(x^k) + \lambda_k P_{C_k}^D[g(x^k) - \alpha D^{-1}F(x^k)] \tag{3.2}$$

Note that, for the case where g is the identity operator and $C_k = C$ for all $k \in N$, our method becomes the relaxed successive approximations scheme given in [9] for fixed points, also described in [5]. Even in this case, our method coincides with an instance of the descent method for VI, given in [10], for an appropriate choice of $\{\lambda_k\}$. An important topic to point out is that our method deals with approximations of the constraint set

C. So, from a practical point of view it is clear the convenience of handling it with approximations that make easier the numerical resolution than the original set C. A second advantage of the method PPM compared to several projection algorithms given in the literature to solve GVI (see for example [14], [15], [17], [18], [19], [20]) is that only one projection and one inversion of operator g is required per iteration, reducing computational costs.

3.3. **Convergence analysis.** Let us consider the following theoretical assumptions on problem (GVI):
A1 . The solution set of GVI, S, is nonempty.
A2 . The operator F is g-cocoercive with respect to *one* solution $\overline{x} \in S$, with modulus $\beta > 0$, that is,

$$\langle F(x) - F(\overline{x}), g(x) - g(\overline{x})\rangle \geq \beta \|F(x) - F(\overline{x})\|^2 \quad \forall \; g(x) \in C. \tag{3.3}$$

Let us observe that **A1** is a standard condition in GVI, see [12],[15], [17]-[20], while **A2** is a usual requirement in the setting of variational inequalities when only one projection is considered ([10], [13], [26]). Assumption **A2** implies that F is g-Lipschitz related to x^* on C, that is, $\|F(x) - F(x^*)\| \leq \frac{1}{\beta}\|g(x) - g(x^*)\|$ for all x with $g(x) \in C$. We point out that condition **A2** does not imply that F is g-monotone on C, we ilustrate this situation by the following example.

Example 3.1. We consider the GVI problem in $I\!R$ given by $F(x) = e^x \sin x$, $g(x) = 8x$ and $C = [0, 8\pi]$. $S = \{0, \pi\}$ is the solution set of GVI, F is g-cocoercive modulus $\beta = 1$ related to $x^* = 0$, F is not g-monotone on C (take for instance $x_1 = 2.8, x_2 = 2.6$).

Let $\{x^k\}$ be a sequence generated by algorithm PPM. If all the iterates are the same starting from some point of the sequence, we show that the last serious step is a solution of problem GVI. In the other case, under condition **A1** and **A2** we prove that the sequence of the g-iterates $\{g(x^k)\}$ converges to $g(\hat{x})$ for some $\hat{x}$. If, in addition, we assume that F is g-Lipschitz on C we conclude that $\hat{x}$ is a solution of GVI. We also show that any accumulation point of the sequence $\{x^k\}$ is a solution of GVI. Furthermore, we guarantee the existence of accumulation points of $\{x^k\}$, if the point-to-set application g^{-1} is locally bounded. Finally we obtain the convergence to a solution of the whole sequence under a stronger requirement, corresponding to the usual condition on g, the existence and continuity of its inverse function.

The following property is used in the proof of our main results. It establishes a g-nonexpansivity condition related to a point of operators T and T_k for all $k \in N$.

Proposition 3.1. *Let F be g-cocoercive with respect to $y \in I\!R^n$ with modulus β and $\alpha \in [0, 2\beta\lambda_{min}(D)]$, where $\lambda_{min}(D)$ is the minimum eigenvalue of D. Then, it holds*

$$\|T_k(x) - T_k(y)\|_D \leq \|g(x) - g(y)\|_D \quad \forall \; g(x) \in C$$

and

$$\|T(x) - T(y)\|_D \leq \|g(x) - g(y)\|_D \quad \forall \; g(x) \in C.$$

Proof. By the definition of T_k and the nonexpansivity condition of the operator P_{C_k} we have that

$$
\begin{aligned}
\|T_k(x) - T_k(y)\|_D^2 &= \|P_{C_k}^D[g(x) - \alpha D^{-1}F(x)] - P_{C_k}^D[g(y) - \alpha D^{-1}F(y)]\|_D^2 \\
&\le \|g(x) - g(y) - \alpha D^{-1}(F(x) - F(y))\|_D^2 \\
&= \|g(x) - g(y)\|_D^2 - 2\alpha\langle g(x) - g(y), F(x) - F(y)\rangle \\
&+ \alpha^2\langle F(x) - F(y), D^{-1}(F(x) - F(y))\rangle
\end{aligned} \tag{3.4}
$$

Since F is g-cocoercive related to y and D^{-1} is a symmetric positive definite matrix, from (3.4) we get that

$$\|T_k(x) - T_k(y)\|_D^2 \le \|g(x) - g(y)\|_D^2 + (\alpha^2\lambda_{max}(D^{-1}) - 2\alpha\beta)\|F(x) - F(y)\|^2. \tag{3.5}$$

Thus, by taking $\alpha \in [0, 2\beta\lambda_{min}(D)]$ we conclude the first inequality of the proposition. In a similar way we obtain the second one. □

The first convergence result concerns the possibility of algorithm PPM to generate a sequence $\{x^k\}$ defined by null steps starting from some iteration.

Theorem 3.2. *Let $\{x^k\}_{k\in\mathbb{N}}$ be a sequence generated by algorithm PPM such that $x^k = \overline{x}$ for all $k \ge \overline{k} \in \mathbb{N}$ and for some $\overline{x} \in \mathbb{R}^n$. If conditions* **R1** *and* **R2** *hold, then, $\overline{x}$ is a solution of GVI.*

Proof. Since $x^{k+1} = x^k = \overline{x}$ for all $k \ge \overline{k}$ we have that $g(x^k) = T_k(x^k)$ and $g(\overline{x}) \in C_k$ for all $k \ge \overline{k}$. It follows that

$$g(\overline{x}) = P_{C_k}^D[g(\overline{x}) - \alpha D^{-1}F(\overline{x})] \quad \forall\ k \ge \overline{k}. \tag{3.6}$$

Hence, by Lemma 2.1 it results that

$$\langle F(\overline{x}), g(y) - g(\overline{x})\rangle \ge 0 \quad \forall\, g(y) \in C_k \quad \forall\ k \ge \overline{k}. \tag{3.7}$$

Now, we show that $\overline{x}$ is a solution of GVI. Indeed, let $y \in \mathbb{R}^n$ such that $g(y) \in C$. By Proposition 2.3 we know that for each $k \in N$ there is $g^k \in C_k$ such that $\{g^k\}$ converges to $g(y)$. Furthermore, we have that $g^k \in C_k \subset C \subset Im(g)$, so, there exists $y^k \in \mathbb{R}^n$ with $g^k = g(y^k)$ for all $k \in N$. This last relation and (3.7) imply that

$$\langle F(\overline{x}), g(y^k) - g(\overline{x})\rangle \ge 0 \quad \forall\ k \ge \overline{k}. \tag{3.8}$$

Passing onto the limit we get that

$$\langle F(\overline{x}), g(y) - g(\overline{x})\rangle \ge 0 \quad \forall\ g(y) \in C. \tag{3.9}$$

Therefore, $\overline{x} \in S$. The proof is completed. □

From now on, we assume that the sequence $\{x^k\}_{k\in\mathbb{N}}$ generated by PPM has an infinite subsequence $\{x^j\}_{j\in\overline{N}}$ of serious steps, that is, $\|R_j(x^j)\|_D > 0$ for all $j \in \overline{N}$. We begin the convergence analysis in this case with a well known property (see for example [1]) that we will use in the next theorem.

Lemma 3.3. *Let $\{\delta_k\}$ and $\{\gamma_k\}$ be nonnegative sequences satisfying the following conditions:*
(i) $\sum_{k=0}^{\infty}\delta_k < +\infty$,
(ii) $\gamma_{k+1} \le \gamma_k + \delta_k \quad \forall\ k \in \mathbb{N}$.
Then, γ_k is a convergent sequence.

Theorem 3.4. *Assume that requirements* **R1**, **R2**, **A1** *and* **A2** *(related to $x^* \in S$) hold. If, in addition, the following data conditions are verified:*

R3 . $\sum_{k=0}^{+\infty} \lambda_k d_\gamma^D(C_k, C) < +\infty \quad \forall \ \gamma \geq 0$,

R4 . $\alpha \in [0, 2\beta\lambda_{min}(D)]$.

Then, the sequence $\{\|g(x^k) - g(x^*)\|_D\}$ is convergent.

Proof. Let x^* be a solution of GVI verifying **A2**. Then, we have $g(x^*) = T(x^*)$. Take $g(x^*) = (1-\lambda_k)g(x^*) + \lambda_k T(x^*)$. By the iterative step of PPM we obtain

$$\begin{aligned} \|g(x^{k+1}) - g(x^*)\|_D &= \|(1-\lambda_k)(g(x^k) - g(x^*)) + \lambda_k(T_k(x^k) - T(x^*))\|_D \\ &\leq (1-\lambda_k)\|g(x^k) - g(x^*)\|_D + \lambda_k\|T_k(x^k) - T_k(x^*)\|_D \\ &+ \lambda_k\|T_k(x^*) - T(x^*)\|_D. \end{aligned} \tag{3.10}$$

Using in this inequality Proposition 3.1 and Remark 3.1 for $\gamma^* \geq \|g(x^*) - \alpha D^{-1}F(x^*)\|_D$ it yields

$$\begin{aligned} \|g(x^{k+1}) - g(x^*)\|_D &\leq \|g(x^k) - g(x^*)\|_D + \lambda_k\|T_k(x^*) - T(x^*)\|_D \\ &\leq \|g(x^k) - g(x^*)\|_D + \lambda_k d_{\gamma^*}^D(C_k, C). \end{aligned}$$

Combining this inequality with Lemma 3.3 for $\gamma_k = \|g(x^k) - g(x^*)\|_D$ and $\delta_k = \lambda_k d_{\gamma^*}^D(C_k, C)$ we obtain the desired conclusion. □

We present now the following basic theorem.

Theorem 3.5. *Assume that the hypotheses of Theorem 3.4 hold. If, in addition, the following condition is satisfied*

R5 . $\sum_{k=0}^{+\infty} \lambda_k(1-\lambda_k) = +\infty$

Then,

$$\lim \inf_{k \longrightarrow +\infty} \|R_k(x^k)\|_D = 0. \tag{3.11}$$

Proof. We start by showing that the sequence $\{g(x^k) - \alpha D^{-1}F(x^k)\}$ is bounded. Indeed, by the theorem above we can consider a bound $L > 0$ of $\{\|g(x^{k+1}) - g(x^*)\|_D\}$. From **A2** we have that $\|F(x^k) - F(x^*)\| \leq \frac{1}{\beta}\|g(x^k) - g(x^*)\|$, so, it follows that $\{g(x^k)\}$ and $\{F(x^k)\}$ are bounded. Hence, $\{g(x^k) - \alpha D^{-1}F(x^k)\}$ is bounded. Furthermore, using the nonexpansivity of P_C^D, we obtain that $\{T(x^k)\}$ is also bounded. Take $\overline{\gamma} > 0$ such that

$$\|g(x^k) - \alpha D^{-1}F(x^k)\|_D \leq \overline{\gamma} \quad \forall \ k \in I\!N.$$

Now, we proceed to prove (3.11) by contradiction. Suppose that the inf $\{\|R_k(x^k)\|_D\} = \overline{\epsilon} > 0$. Let us consider $\epsilon \in (0, \overline{\epsilon})$. Thus, we have

$$\|g(x^k) - T_k(x^k)\|_D = \|R_k(x^k)\|_D > \epsilon \quad \forall \ k \in I\!N.$$

Expanding the square of the D-norm and using the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \epsilon^2 &< \|g(x^k) - T(x^k) + T(x^k) - T_k(x^k)\|_D^2 \\ &\leq \|g(x^k) - T(x^k)\|_D^2 + 2\|g(x^k) - T(x^k)\|_D\|T(x^k) - T_k(x^k)\|_D + \|T(x^k) - T_k(x^k)\|_D^2. \end{aligned} \tag{3.12}$$

Using Remark 3.1 in the above inequality it follows that

$$\epsilon^2 < \|g(x^k) - T(x^k)\|_D^2 + 2\|g(x^k) - T(x^k)\|_D d_{\overline{\gamma}}^D(C_k, C) + (d_{\overline{\gamma}}^D(C_k, C))^2. \tag{3.13}$$

By Lemma 2.4, we get that $\{d^D_{\overline{\gamma}}(C_k, C)\}$ goes to zero. Considering that $\{g(x^k) - T(x^k)\}$ is bounded and $\epsilon < \overline{\epsilon}$, inequality (3.13) implies that there exist $\overline{k} \in I\!N$ such that

$$\|g(x^k) - T(x^k)\|_D > \epsilon \quad \forall \;\; k \geq \overline{k}. \tag{3.14}$$

By considering the same argument used to obtain (3.10), by equality (2.2) and by definition of e^k we obtain

(3.15)
$$\begin{array}{rcl}\|g(x^{k+1}) - g(x^*)\|_D^2 & = & \|(1-\lambda_k)(g(x^k) - g(x^*)) + \lambda_k(T(x^k) - T(x^*)) + \lambda_k(T_k(x^k) - T(x^k))\|_D^2 \\ & = & \|(1-\lambda_k)(g(x^k) - g(x^*) + \lambda_k e^k) + \lambda_k(T(x^k) - T(x^*) + \lambda_k e^k)\|_D^2 \\ & = & (1-\lambda_k)\|g(x^k) - g(x^*) + \lambda_k e^k\|_D^2 + \lambda_k\|T(x^k) - T(x^*) + \lambda_k e^k\|_D^2 \\ & - & \lambda_k(1-\lambda_k)\|g(x^k) - T(x^k)\|_D^2.\end{array}$$

Expanding the square of de D-norm in the equality above and applying the Cauchy-Schwartz inequality, we get

$$\begin{array}{rcl}\|g(x^{k+1}) - g(x^*)\|_D^2 & \leq & (1-\lambda_k)[\|g(x^k) - g(x^*)\|_D^2 + 2\|g(x^k) - g(x^*)\|_D\|\lambda_k e^k\|_D \\ & + & \|\lambda_k e^k\|_D^2] + \lambda_k[\|T(x^k) - T(x^*)\|_D^2 \\ & + & 2\|T(x^k) - T(x^*)\|_D\|\lambda_k e^k\|_D \\ & + & \|\lambda_k e^k\|_D^2] - \lambda_k(1-\lambda_k)\|g(x^k) - T(x^k)\|_D^2,\end{array}$$

which together with Proposition 3.1 gives

(3.16)
$$\begin{array}{rcl}\|g(x^{k+1}) - g(x^*)\|_D^2 & \leq & \|g(x^k) - g(x^*)\|_D^2 + 2\|g(x^k) - g(x^*)\|_D\|\lambda_k e^k\|_D + \|\lambda_k e^k\|_D^2 \\ & - & \lambda_k(1-\lambda_k)\|g(x^k) - T(x^k)\|_D^2.\end{array}$$

Thus, combining this last inequality with Remark 3.1, using the bound L of $\{\|g(x^k) - g(x^*)\|_D\}$ and (3.1) it follows that

$$\begin{array}{rcl}\|g(x^{k+1}) - g(x^*)\|_D^2 & \leq & \|g(x^k) - g(x^*)\|_D^2 + 2L\lambda_k d^D_{\overline{\gamma}}(C_k, C) + (\lambda_k d^D_{\overline{\gamma}}(C_k, C))^2 \\ & - & \lambda_k(1-\lambda_k)\|g(x^k) - T(x^k)\|_D^2.\end{array}$$

Taking into account inequality above, (3.14) and setting

$$\sigma_k :\; = 2L\lambda_k d^D_{\overline{\gamma}}(C_k, C) + (\lambda_k d^D_{\overline{\gamma}}(C_k, C))^2,$$

it results

$$\begin{array}{rcl}\lambda_k(1-\lambda_k)\epsilon^2 & \leq & \lambda_k(1-\lambda_k)\|g(x^k) - T(x^k)\|_D^2 \\ & \leq & \|g(x^k) - g(x^*)\|_D^2 - \|g(x^{k+1}) - g(x^*)\|_D^2 + \sigma_k,\end{array} \tag{3.17}$$

for all $k \geq \overline{k}$. Now, we sum these inequalities for $k = \overline{k}, \overline{k}+1, \overline{k}+2, ..., \overline{k}+m$ to get

(3.18)
$$\epsilon^2 \sum_{k=\overline{k}}^{\overline{k}+m} \lambda_k(1-\lambda_k) \;\; \leq \;\; \|g(x^{\overline{k}}) - g(x^*)\|_D^2 - \|g(x^{\overline{k}+m+1}) - g(x^*)\|_D^2 + \sum_{k=\overline{k}}^{\overline{k}+m} \sigma_k$$

Observe that the condition $\sum_{k=0}^{+\infty} \lambda_k d^D_{\overline{\gamma}}(C_k, C) < +\infty$ implies that

$$\sum_{k=0}^{+\infty} (\lambda_k d^D_{\overline{\gamma}}(C_k, C))^2 < +\infty,$$

so, we obtain that $\sum_{k=\overline{k}}^{+\infty} \sigma_k < \infty$. Hence, must be $\sum_{k=0}^{+\infty} \lambda_k(1-\lambda_k) < +\infty$, which is a contradiction. □

Remark 3.3. This is a fundamental theorem that allows us to obtain convergence results. Indeed, Theorem 3.5 implies that there exists a subsequence of $\{x^k\}$, denoted by $\{x^j\}_{j\in\overline{N}}$, from now on, such that its residues sequence $\{R_j(x^j)\}_{j\in\overline{N}}$ converges to zero. Let us note that this convergent subsequence is easy to be identified from a practical point of view.

We start with the following convergence result.

Lemma 3.6. *Suppose that the hypotheses considered in Theorem 3.5 hold. Then, there is a g-feasible point* $\hat{x}$, $g(\hat{x}) \in C$, *such that the subsequence* $\{g(x^j)\}_{j\in\overline{N}}$ *converges to* $g(\hat{x})$.

Proof. Let $\{x^j\}_{j\in\overline{N}}$ such that $\lim_{j\to+\infty} R_j(x^j) = 0$. Since $\{g(x^k)\}$ is bounded there exists a subsequence $\{x^j\}_{j\in\hat{N}}$ of $\{x^j\}_{j\in\overline{N}}$ such that $\{g(x^j)\}_{j\in\hat{N}}$ converges to some $\hat{g}$. Our conclusion follows that

$$\{g(x^j)\}_{j\in\overline{N}} \subset C \Rightarrow \hat{g} \in C \Rightarrow \exists\, \hat{x} : \hat{g} = g(\hat{x}).$$

□

We need another condition on F to assures that $\overline{x}$ is a solution of GVI. Actually, we have the following property.

Theorem 3.7. *Suppose that hypotheses of Theorem 3.5 are satisfied. If, in addition,* F *is* g*-Lipschitz on* C*, then, there exists a solution* $\overline{x}$ *of GVI such that* $\{g(x^j)\}_{j\in\overline{N}}$ *converges to* $g(\overline{x}) \in C$.

Proof. We consider the sequence $\{x^j\}_{j\in\overline{N}}$ such that $\{g(x^j)\}$ converges to $g(\overline{x})$ for some $\overline{x} \in I\!R^n$, and

$$\lim_{j\to\infty} R_j(x^j) = 0. \tag{3.19}$$

Thus, by the g-Lipschitz condition of F we deduce that the sequence $\{F(x^j)\}$ converges to $F(\overline{x})$, and by Proposition 2.5 it results

$$\begin{aligned} \lim_{j\to\infty} R_j(x^j) &= \lim_{j\to\infty}[g(x^j) - P^D_{C_j}[g(x^j) - \alpha D^{-1}F(x^j)]] \\ &= g(\overline{x}) - P^D_C[g(\overline{x}) - \alpha D^{-1}F(\overline{x})] \end{aligned} \tag{3.20}$$

Combining (3.19) and (3.20) we conclude that $\overline{x}$ is a solution of GVI.

□

We consider again Example 3.1. We observe that F is g-Lipschitz on C. Indeed, by the mean value theorem applied to F it results, $\|F(x) - F(y)\| = \|F'(\xi)\|.\|x - y\| \leq \frac{M}{8}\|g(x) - g(y)\|$ forall $g(x), g(y) \in C$. Then, by Theorem 3.7, there is a sequence $\{g(x^j)\}$ converging to $g(\overline{x})$, $\overline{x} \in \{0, \pi\}$, where $\overline{x} \in \{0, \pi\}$ is a solution of GVI.

The following proposition establishes that any accumulation point of the sequence $\{x^j\}_{j\in\overline{N}}$ is a solution.

Lemma 3.8. *Suppose that the hypotheses considered in Theorem 3.5 hold. If a point* $\overline{x}$ *is an accumulation point of the sequence* $\{x^j\}_{j\in\overline{N}}$ *then,* $\overline{x}$ *is a solution of GVI.*

Proof. Let $\overline{x}$ be a limit point of $\{x^j\}_{j\in\overline{N}}$. So, there is a subsequence that we denote again by $\{x^j\}_{j\in\overline{N}}$ converging to $\overline{x}$.

By the continuity of g and F we get that $\lim_{j\to+\infty} g(x^j) = g(\overline{x}) \in C$, and $\lim_{j\to+\infty} F(x^j) = F(\overline{x})$.

Therefore, by Proposition 2.5 it results

$$\|R(\overline{x})\| = \lim_{j\to+\infty} \|g(x^j) - P^D_{C_j}[g(x^j) - \alpha D^{-1}F(x^j)]\| = 0. \tag{3.21}$$

Hence, $\overline{x}$ is a solution of GVI. □

Now, we obtain the first convergence proposition related to $\{x^k\}$ by considering a local boundary condition on the point-to-set application g^{-1} on C, that is, it carries bounded subsets of C into bounded sets.

Theorem 3.9. *Suppose that the hypotheses considered in Theorem 3.5 hold. If the point-set mapping $g^{-1}\colon I\!\!R^n \longrightarrow \mathcal{P}(I\!\!R^n)$ is locally bounded on C, then, accumulation point set of $\{x^j\}_{j\in\overline{N}}$ is nonempty and it is contained in S.*

Proof. Since $\{g(x^j)\}_{j\in\overline{N}}$ is bounded and g^{-1} is locally bounded, it follows that the sequence $\{x^j\}_{j\in\overline{N}}$ is bounded, hence it has accumulation points. The desired conclusion follows from Lemma 3.8. □

We can illustrate this theorem by observing that the GVI problem defined by $F(x) = x^2$, $g(x) = x^4 + 1$ and $C = [2, +\infty)$ verifies all the conditions of Theorem 3.9. The function F is g-cocoercive with modulus $\beta = 1$ and g^{-1} is not a function, $g^{-1}(y) = \{-\sqrt[4]{y-1}, \sqrt[4]{y-1}\}$ for every $y \in C$.

In order, to obtain stronger results on $\{x^k\}$ and $\{g(x^k)\}$ we demand a stronger condition on F.

Theorem 3.10. *Assume that the hypotheses of Theorem 3.9 and that the condition* **A2** *is verified with respect to each solution of GVI. Then, the whole sequence $\{g(x^k)\}$ converges to $g(\overline{x})$ where $\overline{x}$ is a solution of GVI. Moreover, if the point-set application $g^{-1}\colon I\!\!R^n \longrightarrow \mathcal{P}(I\!\!R^n)$ is locally bounded on C then, every limit point of the whole sequence $\{x^k\}$ is a solution of the original problem.*

Proof. By the theorem above, there is a subsequence of $\{x^j\}_{j\in\overline{N}}$, denoted again by $\{x^j\}$, such that it converges to a solution $\overline{x}$. So, we have

$$\lim_{j\to+\infty} \|g(x^j) - g(\overline{x})\| = 0. \tag{3.22}$$

On the other hand, we can apply Theorem 3.4 to obtain that the whole sequence $\{\|g(x^k) - g(\overline{x})\|_D\}$ is convergent, since $\overline{x} \in S$. Therefore, it must be

$$\lim_{k\to+\infty} \|g(x^k) - g(\overline{x})\| = 0. \tag{3.23}$$

So, we conclude that the whole sequence $\{g(x^k)\}$ converges to $g(\overline{x})$.

Now, we prove that any accumulation point of $\{x^k\}$ belongs to S. Let $\hat{x}$ be a limit point of a subsequence $\{x^i\}$ of $\{x^k\}$. Then, by the continuity of g and the first part of the proof, we get that

$$\lim_{i\to+\infty} g(x^i) = g(\hat{x}) = g(\overline{x}).$$

Now, by considering **A2** with respect to $\overline{x} \in S$ and taking $x = \hat{x}$ in (3.3) we get that $F(\hat{x}) = F(\overline{x})$. Therefore, $R(\hat{x}) = R(\overline{x}) = 0$. Hence $\hat{x}$ is a solution of GVI. The proof is completed. □

Observe that in Example 3.1, $\overline{x} = \pi$ is a solution of GVI, but **A2** does not hold for this point.

Finally, we obtain the convergence of the whole sequence $\{x^k\}$ to a solution by considering a stronger requirement on g, namely, the existence and continuity of the inverse function. This condition is considered in [20] and we find a stronger condition (g is non-singular) in [19]. A similar requirement is also used by He [12].

Theorem 3.11. *Consider the hypotheses of Theorem 3.5 and* **A2** *are verified with respect to each $x \in S$. If g^{-1} is a continuous function then, the sequence $\{x^k\}$ generated by algorithm PPM converges to a solution of GVI.*

Proof. Using an argument similar that in Theorem 3.11, we obtain that there exists $\overline{x} \in S$ such that $\{g(x^k)\}_{k\in I\!\!N}$ converges to $g(\overline{x})$. Due to the continuity of g^{-1} we conclude that

$$\lim_{k\to+\infty} x^k = \overline{x} \in S.$$

The proof is completed.

□

We note that the existence of the inverse function g^{-1} is a strong condition.

4. Preliminary numerical experience

In this section we implement the Perturbed Projection Method defined in Section 3. In order to have a feeling of the behavior of our proposed approach, we tested it on a selection of test problems. The first problem illustrates the performance of algorithm PPM by considering internal polyhedral approximations of the constraint set. The aim of the $2^{\underline{nd}}$ example is to apply PPM to a nonmonotone problem. The other test problems are given in the literature and they have simple constraint sets. For these problems, we present a comparison of our method with the algorithms considered in He [12] and Noor et.al. [19]. The algorithm given by He, is based in only one projection, it considers the sequence $\{(g+F)(x^k)\}$ instead of $\{g(x^k)\}$. The scheme given in [19], is based in two projections and a linear search.

All numerical experiences were performed on a IBM Pentium II with Windows 98 installed and the source code is written in MATLAB 6.12. We consider the following stop conditions:
i) $\|R_\rho(x)\| = \|g(x) - P_C[g(x) - \rho F(x)]\| < 10^{-8}$;
ii) Numbers of iterations (Iter.) equal 1000.
If the iterative process stops by condition i) we present the number of iterations, otherwise, we consider the residual norm $\|R_\rho(x^{1000})\|$.

We denote by Alg. H for the implicit method given by He and Alg. N. for the double projection scheme introduced by Noor et.al.

Now, for sake of completeness, we present the main iteration of algorithms used in our comparisons.

Iteration of Exact Implicit Method (He, [12])

Parameters : $\gamma \in (0,2)$, $D \in I\!R^n \times I\!R^n$, symmetric definite positive matrix.

If $\|R_1(x^k)\| \neq 0$, x^{k+1} solves

$$(g+F)(x) = (g+F)(x^k) + \gamma\delta(x^k)D^{-1}R_1(x^k)$$

where $\delta(x) = \frac{\|R_1(x)\|^2}{\langle R_1(x), D^{-1}R_1(x)\rangle}$.

Iteration of Double Projection Method (Noor et.al., [19])

Parameters : $\sigma, \gamma \in (0,1)$, $\rho \in (0,+\infty)$.

If $\|R_\rho(x^k)\| \neq 0$, by considering a linear search, we obtain y^k: $g(y) = g(x^k) - \eta_k R_\rho(x^k)$ $(\eta_k = \gamma^{m_k})$ such that

$$\rho\langle F(x^k) - F(y^k), R_\rho(x^k)\rangle \leq \sigma\|R_\rho(x^k)\|^2$$

x^{k+1} solves

$$g(x) = P_C[g(x^k) - \alpha_k d^k]$$

where $\quad d^k = \eta_k R_\rho(x^k) + \eta_k F(x^k) + \rho F(y^k) \quad$ and $\quad \alpha_k = \frac{(1-\sigma)\eta_k \|R_\rho(x^k)\|^2}{\|d^k\|^2}$.

Example 4.1. **(Nonlinear GVI with an arbitrary constraint set)** We construct this example in order to analyse the behavior of the algorithm PPM, when the constraint set C is more complex than a ball or a box. We consider a sequence of internal polyhedral approximations of C. The problem is a General Variational Inequality defined by

$$F(x_1, x_2) = (x_2, -x_1), \qquad g(x_1, x_2) = \begin{cases} (x_2^2, -x_1) & \text{if} \qquad x_2 \geq 0 \\ (0, -x_1) & \text{otherwise,} \end{cases}$$

where

$$C = \{(x_1, x_2) \in I\!R^2 \colon 1 \leq x_1 \leq 5,\ (x_1 - 1)^2 \leq x_2 \leq 16\}.$$

Let us note that F is g-cocoercive modulus $\beta = 1$ related to $S = \{(0, 1)\}$.
Exogenous parameters:
Polyhedral approximations $\{C_k\}$: Initially, we consider C_0 like the convex hull of six points belongs to boundary of C. We construct the others sets by increasing the number of involved points. We obtain $\{C_k\} \subset C$ such that $d_\gamma^D(C_k, C) < \frac{1}{k^2+k}, \quad \gamma > 0$.
$D = I$, $\alpha \in (0, 2)$ and $\lambda_k = 1 - \frac{1}{k+1}$;
Initial point: $x^0 = (-9, \sqrt{3})$;
Stop condition: $\|R_k(x^k)\| \leq 10^{-8}$.

α	0.3	0.5	1.0	1.5	1.9
Iter.	61	35	14	14	14

Table 4.1: Example 4.1

Example 4.2. **(Nonmonotone problem)**
This example is important because the operator F is g-cocoercive modulus 1 related to the solution $x^* = (0, ..., 0)$ and nonmonotone on C.

$$F(x_1, ..., x_n) = \sum_{i=1}^{n} \exp{(x_i)} \sin x_i e_i,$$

$$g(x) = Ax = \begin{pmatrix} 8 & 1 & 0 & ... & 0 \\ 1 & 8 & 1 & ... & 0 \\ 0 & 1 & 8 & ... & 0 \\ ... & . & . & ... & . \\ ... & . & . & ... & . \\ ... & . & . & ... & . \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ ... \\ ... \\ ... \\ x_{n-1} \\ x_n \end{pmatrix}$$

and $\quad C = A([0, \pi]^n)$. In this example we consider a fixed $\alpha \in (0, 2]$ and we vary the dimension of the problem.

Dim	10	100	200
Iter	91	98	99

Table 4.2: Example 4.2

Exogenous parameters:
PPM : $C_k \equiv C$, $D = I$, $\lambda_k = 1 - \frac{1}{k+1}$, $\beta = 1, \alpha = 2$;

When Alg.N. is applied to this example (in an heuristic way) the method converges to a solution using three times the number of iterations of algorithm PPM. The Alg.H. is hard to be applied to this example because the inversion of $g + F$ must be considered.

Example 4.3. **(VI problem with ball constraint set)** This example is a classical variational inequality problem $VI(F, C)$ with $F(x) = Hx + c$, where the data is chosen as: $H = VWV$ where $V = I - 2\frac{v.v^t}{\|v\|^2}$ is a Householder matrix and $W = diag(\rho_i)$ with $\rho_i = \cos\frac{i\pi}{n+1} + 1000$. The vectors v and c contains pseudo-random numbers:

$$v_1 = 13846, \quad v_i = (42108v_{i-1} + 13846)mod(46273) \quad i = 2, ..., n$$

$$c_1 = 13846, \quad c_i = (45287c_{i-1} + 13846)mod(46219) \quad i = 2, ..., n$$

For this test problems, the domain set is $C = \{x \in I\!R^n | \|x\| \leq 10^5\}$.

Let us note that this example is similar to such given in [12].

Exogenous parameters:
Alg. H : $D = I$, $\gamma = 1$; Alg. N : $\gamma = 0.8, \rho = 1, \sigma = 0.5$;
Alg. PPM : $C_k \equiv C$, $D = I$, $\beta = 0.0009$, $\alpha = 0.0012$ and $\lambda_k = 1 - \frac{1}{k+1}$;
Initial point $x^0 = (0, .., 0) \in I\!R^n$.

Dimension	Alg. H	Alg. N	Alg. PPM
10	6	73	9
20	6	75	9
50	7	78	9
80	8	81	9
100	9	84	9
200	9	97	9

Table 4.3: Example 4.3

Example 4.4. **(GVI problem with box constraint set)** This example is a general variational inequality problem with $g(x) = Ax + q$ and $F(x) = x$, where

$F(x) = x$, and

$$A = \begin{pmatrix} 4 & -2 & 0 & ... & 0 \\ 1 & 4 & -2 & ... & 0 \\ 0 & 1 & 4 & ... & 0 \\ ... & . & . & ... & . \\ ... & . & . & ... & . \\ ... & . & . & ... & . \\ 0 & 0 & 0 & ... & -2 \\ 0 & 0 & 0 & ... & 4 \end{pmatrix}, \; q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ ... \\ ... \\ ... \\ 1 \\ 1 \end{pmatrix}$$

For this test problems, the domain set $C = \{x \in I\!R^n | 0 \leq x_i \leq 1,\ i = 1, 2, ..., n\}$. We consider this example for different dimensions and related tho three metrics.

Exogenous parameters:
Matrices (symmetric part): $D_1 = I,\ D_2 = J(F \circ g^{-1}) + I,\ D_3 = (I + A)^{-1}$;
Alg. H : $D = D_i$: $i = 1, 2, 3,\ \gamma_1 = 0.5, \gamma_2 = \gamma_3 = 1.9$;
Alg. PPM : $C_k \equiv C,\ D = D_i$: $i = 1, 2, 3,\ \beta = 3,\ \alpha_1 = 3,\ \alpha_2 = 6.9798,\ \alpha_3 = 0.8888$ and $\lambda_k = 1 - \frac{1}{k+1}$;
Initial point $x^0 = -A^{-1}q$.

Dim	Alg. H			Alg.N	PPM		
	D_1	D_2	D_3		D_1	D_2	D_3
10	42	54	30	492	22	12	12
20	42	54	30	489	22	13	12
50	42	54	30	484	22	13	13
80	42	54	31	481	22	13	13
100	42	54	31	480	22	13	13
200	42	54	31	476	13	13	13

Table 4.4: Example 4.4

5. Conclusions

In this paper, we have presented a perturbed projection method, PPM, to solve a general variational inequality problem involving a nonlinear and nonmonotone operator F. Our approach is based on one projection on internal variational approximations of the constraint set related to a D-metric. It is important to handle with approximations of the constraint set C when it doesn't have a special structure like a box or a ball. We can choose α and D to improve the speed of convergence, like in example 4.4. Our convergence analysis requires a weak cocoercivity condition on F. Following [10], the cocoercivity assumption on the whole constraint set is the weakest condition so far to ensure convergence of the simple descent method for nonlinear variational inequality problems. Note that we only require the g-cocoercivity of F related to the solution set. We have obtained new results under a mild assumption on g. We have got the classical convergence result under the usual conditions of existence and continuity of the inverse function g^{-1}.

We have presented preliminary numerical results. We have illustrated the method by considering approximations of the constraint set and a nonmonotone problem. We compare our method with algorithms given in [19] and [12], when the operator F is monotone. Taking into account the number of iterations, the PPM algorithm showed more efficiency than the algorithm given in [19], and it has a behavior similar to the one in the second method. In problems where F is strongly nonlinear with respect to g, like example 3.1, our scheme takes advantages related to the algorithm given in [12].

Let us observe that our method allows variations of parameter α, α_k verifying

$$i)\quad 0 \leq \alpha_k \leq 2\beta\lambda_{min}(D) \quad ii)\quad \exists\ \lim_{k\to+\infty} \alpha_k = \alpha \quad iii)\quad \sum_{k=0}^{\infty} \lambda_k |\alpha_k - \alpha| < +\infty.$$

This modification does not carry numerical advantages in relation with a fixed α.

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Depto de Informática e Estatística - DIE, Centro de Ciências da Natureza - CCN, Universidade Federal do Piaui - UFPI, Teresina, PI, CEP 64049-051 Brasil.
E-mail address: paulo@cos.ufrj.br;Partially supported by PICDT/UFPI-CAPES

Instituto de Matemática - IM, Programa de Engenharia de Sistemas e Computação - PESC/COPPE, Universidade Federal do Rio de Janeiro - UFRJ, CP 68511, Rio de Janeiro, RJ, CEP 21945-970, Brasil
E-mail address: susana@cos.ufrj.br;Partially supported by CNPq, FAPERJ