UNIQUENESS OF THE SOLUTION OF A PARTIAL DIFFERENTIAL EQUATION PROBLEM WITH NON-CONSTANT COEFFICIENTS

ERNESTO PRADO LOPES AND JOSÉ ROBERTO LINHARES DE MATTOS

Abstract. We consider the problem $K(x)u_{xx} = u_t$, $0 < x < 1$, $t \geq 0$, where $K(x)$ is bounded below by a positive constant. The solution on the boundary $x = 0$ is a known function g and $u_x(0,t) = 0$. This is an ill-posed problem in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists. We consider that $1/K(x)$ is Lipschitz and we prove that the existence implies in the uniqueness of the solution $u(x,\cdot) \in L^2(R)$. We use a Wavelet Galerkin Method with the Meyer Multiresolution Analysis.

1. Introduction

We consider the following problem, for $0 < \alpha \leq K\,(x) < +\infty$,

$$
\left\{ \begin{array}{llll} K(x)u_{xx}(x,t) = u_t(x,t) & , \quad t \geq 0 & , & 0 < x < 1 \\ u(0,\cdot) = g & , \quad u_x(0,\cdot) = 0 & & \end{array} \right. \tag{1.1}
$$

We assume that this problem has a solution $u(x,\cdot) \in L^2\,(R)$, for K continuous, and we extend $u(x,t)$ and g to R assuming that both vanish for $t < 0$.

Problem (1.1) is ill-posed in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists. This means that if the solution exists, it does not depend continuously on g (see [2, *p.* 14]).

We consider the Meyer Multiresolution Analysis. The advantage in making use of the Meyer wavelet, is that it has good localization in the frequency domain, since its Fourier transform has compact support. The orthogonal projection onto Meyer scaling spaces, can be considered as a low pass filter, cutting off the high frequencies.

From the variational formulation of the approximating problem on the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. The ill-posedness of the problem is regularized approaching it by well-posed problems (see Theorem 3.4 in [2, *p.* 11]).

We consider that $1/K(x)$ is Lipschitz and we prove that the existence implies the uniqueness of the solution $u(x,\cdot) \in L^2(R)$.

For a function $h \in L^1\,(R) \bigcap L^2\,(R)$ its Fourier Transform is given by $\widehat{h}\,(\xi) := \int_R h\,(x)\; e^{-ix\xi}\,dx$. We use the notations e^x and $\exp x$ indistinctly.

1991 *Mathematics Subject Classification.* 65T60.

2. Uniqueness

The scaling function of the Meyer Multiresolution Analysis is the function φ defined by its Fourier Transform by

$$\widehat{\varphi}\left(\xi\right) := \begin{cases} 1 & , \quad |\xi| \leq \frac{2\pi}{3} \\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\left|\xi\right| - 1\right)\right] & , \quad \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ 0 & , \quad otherwise \end{cases}$$

where ν is a diferentiable function satisfying

$$\nu\left(x\right) = \begin{cases} 0 & se \quad x \leq 0 \\ 1 & se \quad x \geq 1 \end{cases} \tag{2.1}$$

$$\nu\left(x\right) + \nu\left(1-x\right) = 1 \tag{2.2}$$

The associated mother wavelet ψ, called Meyer Wavelet, is given by (see [1])

$$\widehat{\psi}\left(\xi\right) = \begin{cases} e^{\frac{i\xi}{2}}\,\sin\left[\dfrac{\pi}{2}\nu\left(\dfrac{3}{2\pi}\left|\xi\right| - 1\right)\right] & , \quad \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3} \\ e^{\frac{i\xi}{2}}\,\cos\left[\dfrac{\pi}{2}\nu\left(\dfrac{3}{4\pi}\left|\xi\right| - 1\right)\right] & , \quad \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3} \\ 0 & , \quad otherwise \end{cases}$$

We will consider the Meyer Multiresolution Analysis with scaling function φ.

Lemma 1. *The operator $D_j(x)$ defined by*

$$\left[\left(D_j\right)_{lk}(x)\right]_{l\in Z,\ k\in Z} = \left[\frac{1}{K\left(x\right)}\left\langle \varphi'_{jl}\,,\,\varphi_{jk}\right\rangle\right]_{l\in Z,\ k\in Z}$$

satisfies:

1) $\left(D_j\right)_{lk}(x) = -\left(D_j\right)_{kl}(x)$
2) $\left(D_j\right)_{lk}(x) = \left(D_j\right)_{(l-k)\,0}(x)$. *Hence,* $\left(D_j\right)_{lk}(x)$ *are equal along diagonals.*
3) $\|D_j(x)\| \leq \dfrac{\pi 2^{-j}}{K(x)}$

Proof. See Lemma 3.2 in [2, *p.* 6]. □

Let us now consider the following approximating problem[1] in V_j,

$$\begin{cases} K(x)\,u_{xx}(x,t) = P_j u_t(x,t) & , \quad t \geq 0 \quad , \quad 0 < x < 1 \\ u(0,\cdot) = P_j g \\ u_x(0,\cdot) = 0 \\ u(x,t) \in V_j \end{cases} \tag{2.3}$$

Its variational formulation is

$$\begin{cases} \left\langle K\left(x\right)\,u_{xx} - u_t\ ,\ \varphi_{jk}\right\rangle = 0 \\ \left\langle u(0,\cdot)\ ,\ \varphi_{jk}\right\rangle = \left\langle P_j g\,,\,\varphi_{jk}\right\rangle \quad , \quad \left\langle u_x(0,\cdot)\ ,\ \varphi_{jk}\right\rangle = \left\langle 0\,,\,\varphi_{jk}\right\rangle \quad , \quad k \in Z \end{cases}$$

[1]The projection in the first equation of (2.3) is due to the fact that we can have $\varphi \in V_j$ with $\varphi' \notin V_j$ (see [2, *p.* 14]).

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Consider u_j a solution of the approximating problem (2.3), given by $u_j(x,t) = \sum_{l\in Z} w_l(x)\ \varphi_{jl}(t)$. Then, we have $(u_j)_t(x,t) = \sum_{l\in Z} w_l(x)\ \varphi'_{jl}(t)$ and $(u_j)_{xx}(x,t) = \sum_{l\in Z} w''_l(x)\ \varphi_{jl}(t)$. Therefore,

$$K(x)(u_j)_{xx}(x,t) - (u_j)_t(x,t) = K(x)\sum_{l\in Z} w''_l(x)\ \varphi_{jl}(t) - \sum_{l\in Z} w_l(x)\ \varphi'_{jl}(t)$$

Hence

$$\left\langle K(x)(u_j)_{xx} - (u_j)_t\,, \varphi_{jk}\right\rangle = 0 \Longleftrightarrow \left\langle \sum_{l\in Z} K(x)\ w''_l\,\varphi_{jl} - \sum_{l\in Z} w_l\,\varphi'_{jl}, \varphi_{jk}\right\rangle = 0$$

$$\Longleftrightarrow \sum_{l\in Z} K(x)\ w''_l\ \langle \varphi_{jl}\,,\,\varphi_{jk}\rangle = \sum_{l\in Z} w_l\ \langle \varphi'_{jl}\,,\,\varphi_{jk}\rangle$$

$$\Longleftrightarrow K(x)\ w''_k = \sum_{l\in Z} w_l\ \langle \varphi'_{jl}\,,\,\varphi_{jk}\rangle \qquad , \qquad k\in Z$$

$$\Longleftrightarrow \tfrac{d^2}{dx^2}\,w_k = \sum_{l\in Z} w_l\,\frac{1}{K(x)}\ \langle \varphi'_{jl}\,,\,\varphi_{jk}\rangle \ \Longleftrightarrow\ \frac{d^2}{dx^2}\,w_k = \sum_{l\in Z} w_l\ \ (D_j)_{lk}(x)$$

where, as defined before, $(D_j)_{lk}(x) = \frac{1}{K(x)}\left\langle \varphi'_{jl}\,,\,\varphi_{jk}\right\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations:

$$\left\{\begin{array}{l} \tfrac{d^2}{dx^2}\,w = -D_j(x)\,w \\ w(0) = \gamma \\ w'(0) = 0 \end{array}\right. \tag{2.4}$$

where γ is given by

$$P_j g = \sum_{z\in Z} \gamma_z\,\varphi_{jz} = \sum_{z\in Z} \langle g, \varphi_{jz}\rangle\ \varphi_{jz}$$

We will consider (1.1), for the functions $g\in L^2(R)$ such that $\widehat{g}(\cdot)\,.\exp\left(\frac{|\cdot|}{2\alpha}\right)\in L^2(R)$, where $\widehat{g}$ is the Fourier Transform of g. The Inverse Fourier Transform of $\exp\left(-\frac{\xi^2+|\xi|}{2\alpha}\right)$, for instance, satisfies this condition. Define

$$f := \widehat{g}(\cdot)\,.\exp\left(\frac{|\cdot|}{2\alpha}\right)\in L^2(R) \tag{2.5}$$

Theorem 1. *Let u be a solution of the problem (1.1) with condition $u(0,\cdot) = g$ and let f be given by (2.5). Let v_{j-1} be a solution of the problem (2.3) in V_{j-1} for the boundary specification $\widetilde{g}$ such that $\|g-\widetilde{g}\|\le\epsilon$. If $j = j(\epsilon)$ is such that $2^{-j} = \frac{\alpha}{\pi}\log\epsilon^{-1}$ then*

$$\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \epsilon^{1-x^2}\ +\ \|f\|_{L^2(R)}\cdot\epsilon^{\frac{1}{3}(1-x^2)}$$

Proof. See Theorem 3.7 in [2, *p.* 13] □

The infinite-dimensional system of ordinary differential equations (2.4) can be written in the following way:

$$\begin{cases} \frac{dv}{dx} = -D_j(x)w + 0v \\ \frac{dw}{dx} = 0w + v \\ w(0) = \gamma \quad \text{and} \quad v(0) = 0 \end{cases} \qquad \textit{or} \qquad \begin{cases} \frac{dV}{dx} = A_j(x)V \\ V(0) = (0, \gamma) \end{cases}$$

where $V = (v, w) \in X := l^2\,(R)\ X\,l^2\,(R)$, $x \in [0, 1)$ and

$$A_j(x) = \begin{bmatrix} 0 & -D_j\,(x) \\ 1 & 0 \end{bmatrix}$$

with $\|A_j\,(x)\,V\|_X = \|(-D_j\,(x)\,w, v)\|_X = \sqrt{\|D_j\,(x)\,w\|_{l^2}^2 + \|v\|_{l^2}^2}$

Lemma 2. *For all* $j \in Z$, $A_j(x) : X \longrightarrow X$ *is a linear operator bounded uniformly on* $x \in [0, 1)$.

Proof. By Lemma 1 and the hypothesis $0 < \alpha \le K(x) < +\infty$, we have

$$\|D_j(x)\| \le \frac{\pi 2^{-j}}{K(x)} \le \frac{\pi 2^{-j}}{\alpha} := K_j$$

If $\|V\|_X = 1$ then $\|w\|_{l^2} \le 1$ and $\|v\|_{l^2} \le 1$. So,

$$\|A_j\,(x)\,V\|_X = \sqrt{\|D_j\,(x)\,w\|_{l^2}^2 + \|v\|_{l^2}^2} \le \sqrt{K_j^2 + 1}$$

Thus, the operator $A_j(x)$ is bounded uniformly on $x \in [0, 1)$. □

Lemma 3. *If* $\frac{1}{K(x)}$ *is Lipschitz on* $[0, 1)$ *then* $x \longmapsto D_j(x)$ *is Lipschitz on* $[0, 1)$, $\forall\, j \in Z$. *Consequently* $x \longmapsto A_j(x)$ *is Lipschitz on* $[0, 1)$.

Proof. $D_j(x) = \frac{1}{K(x)} B_j(x)$, where $(B_j)_{lk} = \langle \varphi'_{jl}\,, \varphi_{jk}\rangle$. We have $\|B_j\| \le \pi 2^{-j}$ (see proof of the Lemma 3.2 in [2, p. 6]). Then

$$\|D_j(x) - D_j(y)\| \le \left[\frac{1}{K(x)} - \frac{1}{K(y)}\right] \pi 2^{-j} \le L_j\ |x - y|$$

with $L_j = L \cdot \pi 2^{-j}$, where L is the Lipschitz constant of $\frac{1}{K(x)}$.

Now,

$$\begin{aligned} \|A_j(x) - A_j(y)\| &= \sup_{V \in X\ ,\ \|V\|=1} \|(A_j(x) - A_j(y))\,V\|_X \\ &= \sup_{V \in X\ ,\ \|V\|=1} \|(D_j(x) - D_j(y))\,w\|_{l^2} \\ &= \sup_{w \in l^2\ ,\ \|w\|=1} \|(D_j(x) - D_j(y))\,w\|_{l^2} \\ &= \|D_j(x) - D_j(y)\| \\ &\le L_j\,|x - y| \end{aligned}$$

□

Lemma 4. *For all* $j \in Z$, *the operator* $[0,1) \ni x \longmapsto A_j(x)$ *is continuous on operators uniform topology.*

Proof. Let $x \in [0,1)$ and $\epsilon > 0$. By Lemma 3, $A_j(x)$ is Lipschitzian with Lipschitz constant L_j. Let $\delta_\epsilon := \frac{\epsilon}{L_j}$. We have, for $y \in [0,1)$:

$$|x-y| < \delta_\epsilon \Longrightarrow \|A_j(x) - A_j(y)\| \le L_j\,|x-y| < L_j \cdot \delta_\epsilon = \epsilon$$

□

By previous lemmas, we have:

Theorem 2. *The infinite-dimensional system of ordinary differential equations (2.4) has an unique solution.*

Proof. The Lemmas above permit to apply the Theorem 5.1 in $[3, p.\ 127]$. □

Theorem 3. *Let* u *be a solution of the problem (1.1) with condition* $u(0,\cdot) = g$ *where* g *satisfies (2.5). Then for any sequence* j_n, *such that* $j_n \longrightarrow -\infty$ *when* $n \longrightarrow +\infty$, *there exists a unique sequence* u_{j_n} *of solutions of the approximating problems (2.3) in* V_{j_n} *with conditions* $u_{j_n}(0,\cdot) = P_{j_n} g$ *and,* $\forall\, x \in [0,1)$,

$$P_{j_n+1} u_{j_n}(x,\cdot) \longrightarrow u(x,\cdot) \ \text{in } L^2.$$

Proof. From Theorem 1 and Theorem 2, using $\widetilde{g} = g$ and taking into account that the choice of j in Theorem 1, depends only on ϵ and does not depend on u. □

Corollary 1. *The problem (1.1) has at most one solution, for each* $x \in [0,1)$, *where* g *satisfies (2.5).*

3. Conclusion

We had considered solutions $u(x,\cdot) \in L^2(R)$ of the problem $K(x)u_{xx} = u_t$, $0 < x < 1$, $t \ge 0$, with boundary specification $g \in L^2(R)$ and $u_x(0,\cdot) = 0$, where $K(x)$ is bounded below by a positive constant, $\frac{1}{K(x)}$ is Lipschitz and $\widehat{g}(\cdot) . \exp\left(\frac{|\cdot|}{2\alpha}\right) \in L^2(R)$. We had shown that if the solution exists it is unique.

References

[1] Daubechies, I., Ten Lectures on Wavelets, CBMS - NSF ; 61 SIAM, Regional Conferences Series in Applied Mathematics, Pensylvania, USA, 1992.

[2] de Mattos, J.R.L. and Lopes, E.P., A Wavelet Galerkin Method Applied to Partial Differential Equations with Variable Coefficients. Fifth Mississippi State Conference on Differential Equations and Computational Simulations, Electron. J. Diff. Eqns., Conf. 10, 2003, pp.211-225.

[3] Pazzy, Semigroups of Linear Operators and Applications to Partial Differential equations, Applied Mathematical Sciences 44, Springer-Verlag, New York, USA, 1983.

[4] Reginska, T., Sideways Heat Equation and Wavelets, J. Comput. Appl. Math., 63 (1995), pp. 209-214.

[5] Reginska, T. and Eldén, L., Solving the sideways heat equation by a wavelet-Galerkin method, Inverse Problems, 13 (1997), pp. 1093-1106.

[6] Reginska, T., Stability and Convergence of a Wavelet-Galerkin Method for the Sideways Heat Equation, J. Inverse Ill Posed Probl., 8 no 1 (2000), pp. 31-49.

(E. P. Lopes) Federal University of Rio de Janeiro, COPPE, Systems and Computing Engineering Program, Tecnology Center, Bloco H, and Institute of Mathematics, Tecnology Center, Bloco C, Ilha do Fundão, Rio de Janeiro RJ, CEP21945-970, Brazil. E-mail: lopes@cos.ufrj.br

(J. R. L de Mattos) Rural Federal University of Rio de Janeiro, Exact Sciences Institute, Department of Mathematics, BR465 Km 7, Seropédica RJ, CEP 23890-000, Brazil. E-mail: linhares@cos.ufrj.br