A study of local solutions in linear bilevel programming*

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Abstract

In this paper a linear bilevel programming problem (LBP) is considered. Local optimality in LBP is studied by two related problems (P) and $P(M)$. Problem (P) is a one-level model obtained by replacing the innermost problem of LBP by its KKT conditions. Problem $P(M)$ is a penalization of the complementarity constraints of (P) with a penalty parameter M. Characterizations of a (strict) local solution of LBP are derived. In particular, the concept of equilibrium point of $P(M)$ is used to characterize local optima of (P) and LBP. An algorithm to find a local solution of LBP is presented. It uses an equilibrium point procedure which implicitly considers the penalty parameter.

Keywords: Bilevel linear programming, local optimization, exact penalty methods, equilibrium constraints.

1 Introduction

In this work we consider the following linear bilevel program:

$$\begin{array}{lll} \text{(LBP)} & \max_{x,y} & f_1(x,y) = c_1^T x + c_2^T y \\ & \text{s.t.} & x \geq 0, y \text{ solves} \\ & & \quad \begin{array}{ll} \max_y & f_2(x,y) = b^T y \\ \text{s.t.} & A_1 x + A_2 y \leq a \\ & y \geq 0 \end{array} \end{array}$$

where $c_1, x \in \mathcal{R}^{n_1}$, $c_2, b, y \in \mathcal{R}^{n_2}$, $a \in \mathcal{R}^m$, $A_1 \in \mathcal{R}^{m\times n_1}$ and $A_2 \in \mathcal{R}^{m\times n_2}$. This problem has been extensively studied in the literature, see e.g. Refs. 1–6. The LBP with linear constraints in the first level has also been considered, see e.g. Refs. 7, 8. We refer to Ref. 9 for a bibliography survey and to Ref. 10 for more recent results on bilevel and multilevel programming.

Actually, this problem can be reformulated as a mathematical program with equilibrium constraints (MPEC) (Refs. 11–13), since the second level problem can be replaced by a linear complementarity problem. The two formulations are equivalent while considering global solutions, but the equivalence does not hold for local solutions. We are going to show that a local optimum of the MPEC formulation may not give a local optimum of LBP.

Problem LBP belongs to the class of strongly NP-hard problems (Ref. 7). The main difficulties are due to its nonconvexity, which may result in an exponential number of local optima

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(Ref. 14). On the other hand, the design of algorithms has been made difficult due to the lack of computationally attractive theoretical results for the problem.

Our aim is to derive necessary and sufficient conditions for local optimality in problem LBP. Specially, we state a characterization of a local optimum of LBP based on the notion of equilibrium point introduced in Ref. 6. This characterization is particularly useful from a numerical point of view and will be used to devise a local algorithm. Finding local solutions of nonconvex optimization problems is a meaningful deed itself. In addition, local procedures can be used within global algorithms. For problem LBP, this strategy is applied in Refs. 4, 5 for example.

The paper is organized as follows. Section 2 examines the auxiliary problems that will be used in the development. The approach follows that one of Ref. 6. In section 3 we carry out the local analysis of LBP. We derive characterizations for its local and strict local solutions. The most computationally useful characterizations are based on the notion of equilibrium point. In section 4 we present a local algorithm which comprises an equilibrium point procedure.

2 Preliminaries

In this section we consider two auxiliary problems of LBP. The first auxiliary problem (P) is the MPEC obtained by replacing the inner linear problem of LBP by its KKT conditions. The second one $P(M)$ comes from a penalization of the complementarity constraints in (P) with a parameter $M \geq 0$. Thus, we have the following models:

$$\begin{array}{llll} (P) & & P(M) & \\ \max & c_1^T x + c_2^T y & \max & c_1^T x + c_2^T y - M(u^T w + v^T y) \\ \text{s.t.} & A_1 x + A_2 y + w = a, & \text{s.t.} & A_1 x + A_2 y + w = a, \\ & x \geq 0, y \geq 0, w \geq 0, & & x \geq 0, y \geq 0, w \geq 0, \\ & A_2^T u - v = b, & & A_2^T u - v = b, \\ & u \geq 0, v \geq 0, & & u \geq 0, v \geq 0 \\ & u^T w + v^T y = 0 & & \end{array}$$

where $w \in \mathcal{R}^m$ is the primal slack variable and $u \in \mathcal{R}^m$ and $v \in \mathcal{R}^{n_2}$ are dual variables.

Formulations (P) and $P(M)$ are often used as approaches to identify global optima of LBP, see e.g. Refs. 3, 4, 6, 15, 16. These approaches are based on the global equivalence among these problems. Indeed, we show in Ref. 6 that there exists a finite M for which problems (P) and $P(M)$ have the same (empty or nonempty) global solution set which also gives the solution set of LBP.

Here, we use problems (P) and $P(M)$ in the context of local optimality. We derive necessary and sufficient conditions for a local solution (z, s) of (P) to give a local optimum z of LBP. We also present a computationally attractive characterization of the local optima of (P) and LBP by using the notion of equilibrium point of the penalized problem $P(M)$. We show that the penalty parameter can be implicitly consider to get an equilibrium point.

For the sake of convenience, we will be studying local optimality within neighborhoods given by the infinity norm. Let us recall that the infinity norm of $\nu = (\nu_1, \nu_2, \ldots, \nu_p)$ is $\| \nu \|_\infty = \max\{|\nu_i| : 1 \le i \le p\}$ and $B_\varepsilon(\bar{\nu}) = \{\nu \in \mathcal{R}^p : \| \nu - \bar{\nu} \|_\infty \le \varepsilon\}$ is an ε-neighborhood of $\bar{\nu} \in \mathcal{R}^p$. Observe that $B_\varepsilon(\nu, \omega) = B_\varepsilon(\nu) \times B_\varepsilon(\omega)$ for any $(\nu, \omega) \in \mathcal{R}^p \times \mathcal{R}^q$.

We adopt the notation introduced in Ref. 6. We consider the block matrices $A = [A_1\, A_2\, I_m] \in \mathcal{R}^{m \times n}$, $B = [0\ -I_{n_2}\ A_2^T] \in \mathcal{R}^{n_2 \times n}$, $c^T = (c_1^T, c_2^T, 0) \in \mathcal{R}^n$, $z^T = (x^T, y^T, w^T) \in \mathcal{R}^n$ and $s^T = (0, v^T, u^T) \in \mathcal{R}^n$, where $n = n_1 + n_2 + m$, I_k is the $(k \times k)$-identity matrix and 0 is a null matrix of appropriate dimension for each case. We define polyhedra $Z = \{z \in \mathcal{R}_+^n : Az = a\}$ and $S = \{s \in \mathcal{R}_+^n : Bs = b\}$. We denote by X_v the vertex set of a polyhedron X.

Thus, the auxiliary problems are rewritten as:

$$
\begin{array}{llll}
(P) & \max & F(z,s) = c^T z \\
 & \text{s.t.} & z \in Z, s \in S \\
 & & s^T z = 0
\end{array}
\qquad
\begin{array}{lll}
P(M) & \max & F_M(z,s) = c^T z - M s^T z \\
 & \text{s.t.} & z \in Z, s \in S
\end{array}
$$

3 Local Optimality

For our development, it will be useful to consider the point-to-set functions:

$$S(z) = \{s \in S : z^T s = 0\} \quad \text{and} \quad Z(s) = \{z \in Z : s^T z = 0\},$$

which map a point $z \in Z \subset \mathcal{R}^n$ (resp. $s \in S \subset \mathcal{R}^n$) in a polyhedron $S(z) \subset S$ (resp. $Z(s) \subset Z$). These polyhedra have the following property.

Proposition 3.1 *For each $z \in Z$, $S(z)$ is a face of S with vertex set $S_v(z) = S(z) \cap S_v$. For each $s \in S$, $Z(s)$ is a face of Z with vertex set $Z_v(s) = Z(s) \cap Z_v$.*

Proof: Let $z \in Z$. Then, $z \ge 0$. Since $s \ge 0$ for all $s \in S$, we have that $S(z) = \emptyset$ or $S(z)$ is the solution set of the linear program $\min\{z^T s : s \in S\}$. So, it is a face of S. Moreover, the vertices of $S(z)$ are all vertices of S lying in $S(z)$. Similarly, we prove the second part. ■

Functions $S(\cdot)$ and $Z(\cdot)$ are related to our problems as follows. The feasible region of problem (P) is the graph of $S(\cdot)$, $\{(z, s) : z \in Z, s \in S(z)\}$. The feasible set of LBP is the domain of $S(\cdot)$, $\{z \in Z : S(z) \ne \emptyset\}$, or equivalently, the image of $Z(\cdot)$, $\{z \in Z : s^T z = 0 \text{ for some } s \in S\}$. Note that $s \in S(z)$ iff $z \in Z(s)$. Some other relations with the feasible set of LBP are given below.

Proposition 3.2 *For $z \in Z$, the following assertions are equivalent: (i) z is feasible to LBP, (ii) $S_v(z) = S(z) \cap S_v \ne \emptyset$, (iii) $z \in Z(s)$ for some $s \in S_v \subset S$.*

Proof: Let $z \in Z$. Then, z is feasible to LBP iff $S(z) \neq \emptyset$. Since polyhedron $S(z)$ has no lines, $S(z) \neq \emptyset$ iff $S_v(z) \neq \emptyset$. By Proposition 3.1, $S_v(z) = S(z) \cap S_v$. In addition, $s \in S_v(z) = S(z) \cap S_v$ iff $z \in Z(s)$ for $s \in S_v$. Therefore, we get the desired equivalences. ■

As a consequence of the above proposition we obtain a known characterization of the feasible set of LBP in terms of faces of set Z (Ref. 17). Actually, we have the following result.

Corollary 3.1 *The feasible set of LBP is the union of faces of Z. Each of these faces is given by $Z(s)$ for some $s \in S_v$.*

Next we derive some preliminary local properties.

Lemma 3.1 *For each $\bar{z} \in Z$ there is $\varepsilon = \varepsilon(\bar{z}) > 0$ such that $S(z) \subseteq S(\bar{z})$ and $S_v(z) \subseteq S_v(\bar{z})$ for all $z \in Z \cap B_\varepsilon(\bar{z})$. For each $\bar{s} \in S$ there is $\varepsilon = \varepsilon(\bar{s}) > 0$ such that $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for all $s \in S \cap B_\varepsilon(\bar{s})$.*

Proof: We prove the first part. Let $\bar{z} \in Z$. If $\bar{z} = 0$, then $S(z) \subseteq S = S(\bar{z})$ for all $z \in Z$. Now suppose that $\bar{z} \neq 0$. Let us partition the index set $J = \{1, 2, \ldots, n\}$ into the subsets $J_0 = \{j \in J : \bar{z}_j = 0\}$ and $J_1 = J \setminus J_0$. First we prove that there is $\varepsilon > 0$ such that, if $z \in Z \cap B_\varepsilon(\bar{z})$, then $z_j > 0$ for all $j \in J_1$. Since $\bar{z} \neq 0$, it must be $J_1 \neq \emptyset$. For each $j \in J_1$, let us define the set $C_j = \{z \in \mathcal{R}^n : z_j = 0\}$. Then, set C_j is closed, convex and nonempty ($0 \in C_j$). As $\bar{z} \notin C_j$, there is a hyperplane separating $\bar{z}$ and C_j strongly. Then, it is known that $d(\bar{z}, C_j) = \min\{\| z - \bar{z} \|_\infty : z \in C_j\} > 0$. Hence, there exists $\varepsilon \in \mathcal{R}$ such that $0 < \varepsilon < \min\{d(\bar{z}, C_j) : j \in J_1\}$. Let $z \in Z \cap B_\varepsilon(\bar{z}) \neq \emptyset$. Then, $z \notin C_j$ for all $j \in J_1$, that is, $z_j > 0$ for all $j \in J_1$. Now, we prove the inclusion $S(z) \subseteq S(\bar{z})$ for $z \in Z \cap B_\varepsilon(\bar{z})$. We consider two cases. If $S(z) = \emptyset$, trivially $S(z) \subseteq S(\bar{z})$ (it may be $S(\bar{z}) = \emptyset$). Otherwise, let $s \in S(z)$. Then, $s_j = 0$ for all $j \in J_1$. Consequently, $\bar{z}^T s = \sum_{j \in J_1} \bar{z}_j s_j + \sum_{j \in J_0} \bar{z}_j s_j = 0$. Hence, $s \in S(\bar{z})$, implying $S(z) \subseteq S(\bar{z})$. In any case, there exists $\varepsilon > 0$ such that $S(z) \subseteq S(\bar{z})$ for all $z \in Z \cap B_\varepsilon(\bar{z})$. In particular, the inclusion of the extreme points also holds due to $S_v(z) = S(z) \cap S_v$ for any $z \in Z$.

We prove the second part by interchanging the roles of $\bar{z}$ and Z with $\bar{s}$ and S in the previous argument. It follows that there is $\varepsilon > 0$ such that $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for all $s \in S \cap B_\varepsilon(\bar{s})$. ■

Corollary 3.2 *For each $(\bar{z}, \bar{s}) \in Z \times S$ there is $\varepsilon = \varepsilon(\bar{z}, \bar{s}) > 0$ such that $S(z) \subseteq S(\bar{z})$, $S_v(z) \subseteq S_v(\bar{z})$, $Z(s) \subseteq Z(\bar{s})$ and $Z_v(s) \subseteq Z_v(\bar{s})$ for all $(z, s) \in (Z \times S) \cap B_\varepsilon(\bar{z}, \bar{s})$.*

Proof: It follows from Lemma 3.1 by taking $\varepsilon \in \mathcal{R}$ such that $0 < \varepsilon < \min\{\varepsilon(\bar{z}), \varepsilon(\bar{s})\}$. ■

Lemma 3.2 *Let $(\bar{z},\bar{s}) \in Z \times S$. Then, there is $\varepsilon > 0$ such that $s \in S(\bar{z})$ and $z \in Z(\bar{s})$ if $(z,s) \in B_\varepsilon(\bar{z},\bar{s})$ and is feasible to problem (P).*

Proof: Let $(\bar{z},\bar{s}) \in Z \times S$. By Corollary 3.2, there is $\varepsilon > 0$ such that $S(z) \subseteq S(\bar{z})$ and $Z(s) \subseteq Z(\bar{s})$ for all $(z,s) \in (Z \times S) \cap B_\varepsilon(\bar{z},\bar{s})$. Let $(z,s) \in B_\varepsilon(\bar{z},\bar{s})$, feasible to (P). So, $(z,s) \in (Z \times S) \cap B_\varepsilon(\bar{z},\bar{s})$, $s \in S(z)$ and $z \in Z(s)$. Therefore, $s \in S(\bar{z})$ and $z \in Z(\bar{s})$. ■

Now we can characterize a local solution of LBP in terms of a local solution of (P).

Theorem 3.1 *A point $\bar{z}$ is a local solution of LBP if, and only if, $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z},s)$ is a local solution of problem (P) for all vertex $s \in S_v(\bar{z})$.*

Proof: First assume that $\bar{z} \in Z$ is a local solution of LBP. Then, there is $\varepsilon > 0$ such that $c^T z \leq c^T \bar{z}$ for all $z \in B_\varepsilon(\bar{z})$ with $S(z) \neq \emptyset$. By Proposition 3.2, we have $S_v(\bar{z}) \neq \emptyset$. Let $\bar{s} \in S_v(\bar{z})$. So, $(\bar{z},\bar{s})$ is feasible to (P). Let $(z,s) \in B_\varepsilon(\bar{z},\bar{s})$, feasible to (P). Then, $z \in B_\varepsilon(\bar{z})$ and $S(z) \neq \emptyset$ yielding that $c^T z \leq c^T \bar{z}$. Therefore, $(\bar{z},\bar{s})$ is a local solution of (P). Since $\bar{s}$ is an arbitrary element in $S_v(\bar{z})$, we conclude that $(\bar{z},s)$ is a local solution of (P) for all $s \in S_v(\bar{z})$.

Conversely, let $\bar{z} \in Z$ such that $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z},s)$ is a local solution of (P) for all $s \in S_v(\bar{z})$. Suppose, by contradiction, that $\bar{z}$ is not a local solution of LBP. Let $\bar{\varepsilon} = \varepsilon(\bar{z}) > 0$ be given according to Lemma 3.1. Then, there is $\hat{z} \in B_{\bar{\varepsilon}}(\bar{z})$, feasible to LBP, such that $c^T \hat{z} > c^T \bar{z}$ and $S_v(\hat{z}) \subseteq S_v(\bar{z})$. By Proposition 3.2, $S_v(\hat{z}) \neq \emptyset$. Let $\hat{s} \in S_v(\hat{z}) \subseteq S_v(\bar{z})$. Let us define $z(\lambda) = \lambda\hat{z} + (1-\lambda)\bar{z}$. Since $\hat{z}$ and $\bar{z}$ belong to the convex set $Z(\hat{s})$, then $z(\lambda) \in Z(\hat{s})$ for all $\lambda \in [0,1]$. Hence, $(z(\lambda),\hat{s})$ is feasible to (P) for all $\lambda \in [0,1]$. Since $z(\lambda) \to \bar{z}$ as $\lambda \to 0$, for each $\varepsilon > 0$, there is $\lambda \in (0,1]$ such that $(z(\lambda),\hat{s}) \in B_\varepsilon(\bar{z},\hat{s})$ is feasible to (P) and satisfies $c^T z(\lambda) = c^T \bar{z} + \lambda(c^T \hat{z} - c^T \bar{z}) > c^T \bar{z}$. Therefore, there is $\hat{s} \in S_v(\bar{z})$ such that $(\bar{z},\hat{s})$ is not a local solution of (P), which contradicts the hypothesis. Hence, $\bar{z}$ must be a local solution of LBP. ■

Remark 3.1 A point $\bar{z}$ may not be a local solution of problem LBP if $(\bar{z},\bar{s})$ is a local solution of (P) only for some $\bar{s} \in S_v(\bar{z})$. This situation is illustrated by example (E1), figure 1. The feasible set of LBP is shown in bold. Point A is the global solution. Let $\bar{z} = (\bar{x},\bar{y},\bar{w}_1,\bar{w}_2)^T = (1,1,0,0)^T \in Z$, corresponding to point B, and $\bar{s} = (0,\bar{v},\bar{u}_1,\bar{u}_2)^T = (0,0,0,1)^T \in S_v(\bar{z})$. Although $(\bar{z},\bar{s})$ is a local solution of (P), $\bar{z}$ is not a local solution of LBP. On the other hand, for $\hat{s} = (0,0,1,0)^T \in S_v(\bar{z})$, point $(\bar{z},\hat{s})$ is not a local solution of (P).

Remark 3.2 The equivalence given in Theorem 3.1 does not hold in the case of strict local optimality. Moreover, it is possible to have a strict local solution $\bar{z}$ of LBP such that $(\bar{z},s)$ is not a strict local solution of (P) for every $s \in S_v(\bar{z})$. Actually, this situation occurs whenever $S(\bar{z})$ is not a singleton. For instance, let us consider example (E1), figure 1, where we change the first level objective function to $f_1(x,y) = -y$. Now, point B, corresponding to $\bar{z} = (1,1,0,0)^T \in Z$,

$$\begin{array}{lll} \text{(E1)} & \max & f_1(x,y) = -x \\ & \text{s.t.} & x \geq 0,\ y \text{ solves} \\ & & \quad \max \quad f_2(x,y) = -y \\ & & \quad \text{s.t.} \quad x + y \geq 2 \\ & & \qquad\qquad x - y \leq 0 \\ & & \qquad\qquad y \geq 0 \end{array}$$

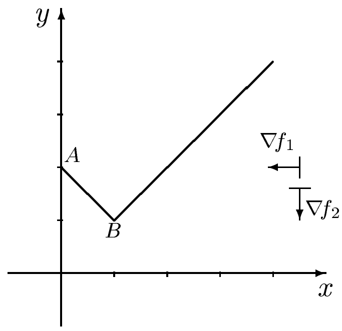

Figure 1:

is a strict local solution of LBP. Actually, it is the global solution. We have that $S_v(\bar{z}) = \{\bar{s} = (0,0,0,1)^T, \hat{s} = (0,0,1,0)^T\}$. Since $s(\alpha) = \alpha\bar{s} + (1-\alpha)\hat{s} \in S(\bar{z})$ for all $\alpha \in [0,1]$, it follows that $(\bar{z}, \bar{s})$ and $(\bar{z}, \hat{s})$ are not strict local solutions of (P). In Corollary 3.4, we are going to establish a characterization for strict local optimality.

Theorem 3.1 suggests searching local solutions of problem LBP among local solutions of (P). These points can be characterized by the penalized problem, bringing about a better computational insight. The characterizations will be stated using the notion of equilibrium of the penalized problem.

Definition 3.1 *A point* $(\bar{z}, \bar{s}) \in Z \times S$ *is an* **equilibrium point** *of the penalized problem* $P(M)$ *if there is* $\bar{M} \geq 0$ *such that, for each* $M \geq \bar{M}$, *it holds*

$$\max\{F_M(\bar{z}, s) : s \in S\} = F_M(\bar{z}, \bar{s}) = \max\{F_M(z, \bar{s}) : z \in Z\}. \tag{1}$$

If the equilibrium point $(\bar{z}, \bar{s})$ *additionally satisfies*

$$\{\bar{z}\} = \arg\max\{F_M(z, \bar{s}) : z \in Z\}, \tag{2}$$

it is called a **primal strict equilibrium point**. *If the equilibrium point* $(\bar{z}, \bar{s})$ *also verifies*

$$\{\bar{s}\} = \arg\max\{F_M(\bar{z}, s) : s \in S\}, \tag{3}$$

it is said to be a **dual strict equilibrium point**. *A primal and dual strict equilibrium point is simply called a* **strict equilibrium point**.

Let us observe that the definition of an equilibrium point of $P(M)$ was introduced in Ref. 18. It was also considered in Ref. 6 for the penalized problem related to a linear bilevel problem with constraints in the first level. The (nonstrict) equilibrium required in Definition 3.1 extends the equilibrium considered by the mountain climbing algorithm given for a bilinear program

(Ref. 19). Indeed, in our case, equality (1) must be fulfilled for all $M \geq \bar{M}$, which means that the equilibrium has to be held by a family of parametric bilinear problems $P(M)$.

In order to characterize local solutions of (P) and LBP, we establish an important property of an equilibrium point.

Lemma 3.3 *If $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem $P(M)$, then*

$$\min\{\bar{z}^T s : s \in S\} = \min\{\bar{s}^T z : z \in Z\} = \bar{s}^T \bar{z} = 0. \tag{4}$$

Proof: Let $(\bar{z}, \bar{s})$ be an equilibrium point. Then, $\min\{\bar{z}^T s : s \in S\} = \bar{z}^T \bar{s}$ comes from the first equality given in (1). Now, suppose, by contradiction, that $\bar{s}^T \bar{z} > \bar{s}^T \hat{z} = \min\{\bar{s}^T z : z \in Z\}$. Take $M > \max\{\bar{M}, c^T(\bar{z} - \hat{z})/\bar{s}^T(\bar{z} - \hat{z})\}$. Thus, $c^T \hat{z} - M\bar{s}^T \hat{z} > c^T \bar{z} - M\bar{s}^T \bar{z}$, which contradicts the second equality in (1). Therefore, $\bar{s}^T \bar{z} = \min\{\bar{s}^T z : z \in Z\}$. Now we show that $\bar{s}^T \bar{z} = 0$. Let $\hat{s} \in \arg\min\{\bar{z}^T s : s \in S_v\}$. Then, $\hat{s} = ((D^{-1}b)^T, 0)^T$, where D is a basis of B. Consider the optimal reduced cost $\hat{z}^T = \bar{z}^T - \bar{z}_D^T D^{-1} B \geq 0$. We have that $A\hat{z} = A\bar{z} = a$ since $AB^T = 0$. Hence, $\hat{z} \in Z$. Using the first two equalities given in (4) and the definitions of $\hat{z}$ and $\hat{s}$, it follows that $0 \leq \bar{z}^T \bar{s} \leq \hat{z}^T \bar{s} = \bar{z}^T \bar{s} - \bar{z}_D^T D^{-1} B\bar{s} = \bar{z}^T \bar{s} - \bar{z}_D^T D^{-1} b = \bar{z}^T \bar{s} - \bar{z}^T \hat{s} = 0$. Then, $\bar{s}^T \bar{z} = 0$. ∎

In particular, for a dual strict equilibrium point we get the following stronger result.

Corollary 3.3 *A point $(\bar{z}, \bar{s})$ is a dual strict equilibrium point of the penalized problem $P(M)$ if, and only if, it is an equilibrium point and $S(\bar{z}) = S_v(\bar{z}) = \{\bar{s}\}$.*

Proof: Assume that $(\bar{z}, \bar{s})$ is a dual strict equilibrium point. Then, it is an equilibrium point. By Lemma 3.3, we have that $\bar{s} \in S(\bar{z})$. Let $\hat{s} \in S(\bar{z})$. Then, $F_{\bar{M}}(\bar{z}, \hat{s}) = F_{\bar{M}}(\bar{z}, \bar{s})$ and so $\hat{s} \in \arg\max\{F_{\bar{M}}(\bar{z}, s) : s \in S\}$. By condition (3), it must be $\hat{s} = \bar{s}$. It results that $S(\bar{z}) = \{\bar{s}\}$. And since $\emptyset \neq S_v(\bar{z}) \subseteq S(\bar{z})$, the desired equalities follow.

Conversely, assume that $(\bar{z}, \bar{s})$ is an equilibrium point and $S(\bar{z}) = S_v(\bar{z}) = \{\bar{s}\}$. Hence, $\bar{z}^T s > 0$ for all $s \in S$. It results that $F_M(\bar{z}, s) - F_M(\bar{z}, \bar{s}) = -M\bar{z}^T s < 0$ for all $s \in S$ and all $M > 0$. Therefore, we get condition (3) and so $(\bar{z}, \bar{s})$ is a dual strict equilibrium point. ∎

Lemma 3.3 says that an equilibrium point is feasible to (P). With this property we now show the equivalences between a local solution of (P) and an equilibrium point of $P(M)$.

Theorem 3.2 *A point $(\bar{z}, \bar{s})$ is a local solution of (P) if, and only if, it is an equilibrium point of the penalized problem $P(M)$.*

Proof: Assume that $(\bar{z}, \bar{s})$ is a local solution of (P). So, there is $\varepsilon > 0$ such that $c^T z \leq c^T \bar{z}$ for all $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P). First, we prove that there is $\bar{M} > 0$ such that $\bar{z}$ is a local maximum of $F_M(\cdot, \bar{s})$ over Z for all $M \geq \bar{M}$. Let us define $\mathcal{Z} = Z \cap B_\varepsilon(\bar{z})$. Since $F_M(\cdot, \bar{s})$ is a linear function for each M and $\mathcal{Z}$ is a compact polyhedron, it results that

$$\max_{z \in \mathcal{Z}} F_M(z, \bar{s}) = \max_{z \in \mathcal{Z}_v} F_M(z, \bar{s}) = \max \left\{ \max_{z \in \mathcal{Z}_v \cap Z(\bar{s})} F_M(z, \bar{s}), \max_{z \in \mathcal{Z}_v \setminus Z(\bar{s})} F_M(z, \bar{s}) \right\} \tag{5}$$

for each M. For every $z \in \mathcal{Z}_v \cap Z(\bar{s})$ we have that $(z, \bar{s}) \in B_\varepsilon(\bar{z}, \bar{s})$ and is feasible to (P). Thus, it follows that

$$F_M(z, \bar{s}) = c^T z \leq c^T \bar{z} = F_M(\bar{z}, \bar{s}) \quad \forall z \in \mathcal{Z}_v \cap Z(\bar{s}), \ \forall M \in \mathcal{R}. \tag{6}$$

Let $M_0 = \sup\{(c^T z - c^T \bar{z})/\bar{s}^T z : z \in \mathcal{Z}_v \setminus Z(\bar{s})\}$. As $\mathcal{Z}_v \setminus Z(\bar{s})$ is a finite set, we have that $M_0 = -\infty$, if $\mathcal{Z}_v \setminus Z(\bar{s}) = \emptyset$, or $M_0 \in \mathcal{R}$, otherwise. Let us consider $\bar{M} > \max\{0, M_0\}$. Then,

$$F_M(z, \bar{s}) = c^T z - M \bar{s}^T z < c^T \bar{z} = F_M(\bar{z}, \bar{s}) \quad \forall z \in \mathcal{Z}_v \setminus Z(\bar{s}), \ \forall M \geq \bar{M} > 0. \tag{7}$$

By (5)-(7) and the fact that $\bar{z} \in \mathcal{Z}$, we conclude that $\max\{F_M(z, \bar{s}) : z \in \mathcal{Z}\} = F_M(\bar{z}, \bar{s})$ for all $M \geq \bar{M}$. Thus, $\bar{z}$ is a local solution of $\max\{F_M(z, \bar{s}) : z \in Z\}$ for all $M \geq \bar{M}$. As any local solution of a linear problem is a global solution, we obtain that $\max\{F_M(z, \bar{s}) : z \in Z\} = F_M(\bar{z}, \bar{s})$ for all $M \geq \bar{M} > 0$. On the other hand, we have that $F_M(\bar{z}, s) - F_M(\bar{z}, \bar{s}) = -M \bar{z}^T s \leq 0$ for all $s \in S$ and all $M \geq \bar{M} > 0$. Therefore, $\max\{F_M(\bar{z}, s) : s \in S\} = F_M(\bar{z}, \bar{s})$ for all $M \geq \bar{M} > 0$. It results that $(\bar{z}, \bar{s})$ is an equilibrium point.

Conversely, assume that $(\bar{z}, \bar{s})$ is an equilibrium point. Then, there is $\bar{M} \geq 0$ such that $c^T z - \bar{M} \bar{s}^T z \leq c^T \bar{z} - \bar{M} \bar{s}^T \bar{z}$ for all $z \in Z$. In addition, by Lemma 3.3, $(\bar{z}, \bar{s})$ is feasible to (P). Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P), with $\varepsilon > 0$ according to Lemma 3.2. Then, $\bar{s}^T z = \bar{s}^T \bar{z} = 0$. The previous expressions imply that $c^T z \leq c^T \bar{z}$. Therefore, $(\bar{z}, \bar{s})$ is a local solution of (P). ■

Let us note that the above result is stronger than Theorem 9 in Ref. 6, which establishes a similar equivalence for the LBP with linear constraints in the first level. Indeed, such a theorem uses the feasibility of $(\bar{z}, \bar{s})$ as a premise. Now, we have shown that the complementarity condition is a consequence of the equilibrium. This fact guarantees that finding an equilibrium point is enough to have a local solution of (P). We establish a similar result for strict local solutions as follows.

Theorem 3.3 *A point $(\bar{z}, \bar{s})$ is a strict local solution of (P) if, and only if, it is a strict equilibrium point of the penalized problem $P(M)$.*

Proof: Assume that $(\bar{z}, \bar{s})$ is a strict local solution of (P) within a neighborhood $B_\varepsilon(\bar{z}, \bar{s})$. By Theorem 3.2, we conclude that $(\bar{z}, \bar{s})$ is an equilibrium point. Let us consider $\hat{s} \in S$ and $\hat{z} \in Z$ such that

$$\max_{s \in S} F_M(\bar{z}, s) = F_M(\bar{z}, \hat{s}) = F_M(\bar{z}, \bar{s}) = F_M(\hat{z}, \bar{s}) = \max_{z \in Z} F_M(z, \bar{s}) \ \ \forall M \geq \bar{M}. \tag{8}$$

We have to show that $\hat{z} = \bar{z}$ and $\hat{s} = \bar{s}$ in order to get conditions (2)-(3). Since $\bar{s}^T\bar{z} = 0$, equalities (8) imply that $\bar{z}^T\hat{s} = \bar{s}^T\hat{z} = 0$ and $c^T\hat{z} = c^T\bar{z}$. Thus, $\hat{s} \in S(\bar{z})$ and $\hat{z} \in Z(\bar{s})$. Since $Z(\bar{s})$ and $S(\bar{z})$ are convex sets, there is $\alpha \in (0,1)$ such that $z(\alpha) \in Z(\bar{s}) \cap B_\varepsilon(\bar{z})$ and $s(\alpha) \in S(\bar{z}) \cap B_\varepsilon(\bar{s})$, where $z(\alpha) = \alpha\hat{z} + (1-\alpha)\bar{z}$ and $s(\alpha) = \alpha\hat{s} + (1-\alpha)\bar{s}$. Hence, $(z(\alpha), \bar{s})$ and $(\bar{z}, s(\alpha))$ belong to $B_\varepsilon(\bar{z}, \bar{s})$ and are feasible to (P). Moreover, $c^T z(\alpha) = c^T\bar{z}$ yielding that $F(z(\alpha), \bar{s}) = F(\bar{z}, s(\alpha)) = F(\bar{z}, \bar{s}) = c^T\bar{z}$. As $(\bar{z}, \bar{s})$ is a strict local solution, it must be $z(\alpha) = \bar{z}$ and $s(\alpha) = \bar{s}$. Therefore, $\hat{z} = \bar{z}$ and $\hat{s} = \bar{s}$. We get conditions (2)-(3) and so $(\bar{z}, \bar{s})$ is a strict equilibrium point.

Conversely, assume that $(\bar{z}, \bar{s})$ is a strict equilibrium point. Let $\varepsilon > 0$ verifying Lemma 3.2 and $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P), with $(z, s) \neq (\bar{z}, \bar{s})$. It follows that $s \in S(\bar{z})$ and $z \in Z(\bar{s})$. By Corollary 3.3, $S(\bar{z}) = \{\bar{s}\}$. Thus, it must be $s = \bar{s}$ and $z \neq \bar{z}$. By condition (2), we have that $c^T z = F_{\bar{M}}(z, \bar{s}) < F_{\bar{M}}(\bar{z}, \bar{s}) = c^T\bar{z}$. Therefore, $(\bar{z}, \bar{s})$ is a strict local solution of (P). ■

A characterization of local solutions of LBP in terms of equilibrium points follows.

Theorem 3.4 *A point $\bar{z}$ is a (strict) local solution of LBP if, and only if, $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a (primal strict) equilibrium point of the penalized problem $P(M)$ for all vertex $s \in S_v(\bar{z})$.*

Proof: The equivalence in the nonstrict case follows by Theorems 3.1 and 3.2.

Now, assume that $\bar{z}$ is a strict local solution of LBP. It remains to show that condition (2) holds for all $s \in S_v(\bar{z}) \neq \emptyset$. Let be $\bar{s} \in S_v(\bar{z})$ arbitrary. Similarly to the proof of Theorem 3.3, consider $\hat{z} \in Z$ such that $\max\{F_M(z, \bar{s}) : z \in Z\} = F_M(\hat{z}, \bar{s}) = F_M(\bar{z}, \bar{s})$ for all $M \geq \bar{M}$. Again, we conclude that there is $\alpha \in (0,1)$ such that $z(\alpha) \in Z(\bar{s}) \cap B_\varepsilon(\bar{z})$ and $c^T z(\alpha) = c^T\bar{z}$, where $z(\alpha) = \alpha\hat{z} + (1-\alpha)\bar{z}$. As $\bar{z}$ is a strict local solution and $z(\alpha)$ is feasible to LBP, it must be $z(\alpha) = \bar{z}$. Hence, $\hat{z} = \bar{z}$ and so condition (2) holds. Therefore, $(\bar{z}, \bar{s})$ is a primal strict equilibrium point. Moreover, since $\bar{s}$ is arbitrary in $S_v(\bar{z})$, we conclude that $(\bar{z}, s)$ is a primal strict equilibrium point for all $s \in S_v(\bar{z})$.

Conversely, let $\bar{z} \in Z$ such that $S_v(\bar{z}) \neq \emptyset$ and $(\bar{z}, s)$ is a primal strict equilibrium point for all $s \in S_v(\bar{z})$. By Lemma 3.1, there is $\varepsilon = \varepsilon(\bar{z}) > 0$ such that $S_v(z) \subseteq S_v(\bar{z})$ for all $z \in Z \cap B_\varepsilon(\bar{z})$. Consider $z \in B_\varepsilon(\bar{z}) \setminus \{\bar{z}\}$, feasible to LBP. By Proposition 3.2, we conclude that there is $\bar{s} \in S_v(z) \subseteq S_v(\bar{z})$. Thus, $(\bar{z}, \bar{s})$ is a primal strict equilibrium point. Using condition (2), it follows that $c^T z = F_M(z, \bar{s}) < F_M(\bar{z}, \bar{s}) = c^T\bar{z}$. Therefore, $\bar{z}$ is a strict local solution of LBP. ■

In Ref. 20 Audet *et al.* reformulate a disjoint linear program (BILD) as a linear maxmin problem (LMM) and show that a local solution of LMM gives a local solution of BILD. They also note that the number of local optima of LMM may be less than that of BILD. These results resemble the statement of Theorem 3.4. In fact, an LMM is a special case of an LBP, $P(M)$

is a parametric disjoint bilinear problem and $(\bar{z}, \bar{s})$ is an equilibrium point iff there is $\bar{M} \geq 0$ such that $(\bar{z}, \bar{s})$ is a local solution of $P(M)$ for all $M \geq \bar{M}$ (Ref. 18). It is also worth noting that Theorem 3.4 goes further than Proposition 6 in Ref. 20 in the sense that it presents a characterization.

The next corollary expresses the equivalence corresponding to Theorem 3.1 for strict optimality. Note that in this case the condition of $S_v(\bar{z})$ being a singleton is necessary to characterize a strict local solution $\bar{z}$ of LBP.

Corollary 3.4 *The following assertions are equivalent: (i) $(\bar{z}, \bar{s})$ is a strict local solution of problem (P), (ii) $(\bar{z}, \bar{s})$ is a strict equilibrium point of the penalized problem $P(M)$, (iii) $\bar{z}$ is a strict local solution of problem LBP and $S_v(\bar{z}) = \{\bar{s}\}$.*

Proof: By Corollary 3.3, we conclude that $(\bar{z}, \bar{s})$ is a strict equilibrium point iff $(\bar{z}, \bar{s})$ is a primal strict equilibrium point and $S_v(\bar{z}) = \{\bar{s}\}$. Thus, the result follows by Theorems 3.3 and 3.4. ∎

4 Equilibrium points

In this section we explore the characterization of local solutions of LBP in terms of equilibrium points as given by Theorem 3.4. We aim to find results which are more attractive from a computational point of view. We start giving a characterization for a equilibrium point which does not depend on parameter M.

Theorem 4.1 *A point $(\bar{z}, \bar{s})$ is an equilibrium point of the penalized problem $P(M)$ if, and only if, $\bar{s} \in S$ and $\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}$.*

Proof: Let $(\bar{z}, \bar{s})$ be an equilibrium point of $P(M)$. Thus, $(\bar{z}, \bar{s}) \in Z \times S$. By Lemma 3.3, $\bar{s}^T \bar{z} = 0$, that is, $\bar{z} \in Z(\bar{s})$. Therefore, we have that $c^T \bar{z} \leq \max\{c^T z : z \in Z(\bar{s})\} \leq \max\{c^T z - \bar{M} \bar{s}^T z : z \in Z\} = c^T \bar{z}$, where $\bar{M} \geq 0$ is given by Definition 3.1. Hence, $\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}$.

Conversely, assume that $\bar{s} \in S$ and $\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}$. Then, $(\bar{z}, \bar{s})$ is feasible to (P). By Lemma 3.1, there is $\varepsilon = \varepsilon(\bar{s})$ such that $Z(s) \subseteq Z(\bar{s})$ for all $s \in S \cap B_\varepsilon(\bar{s})$. Let $(z, s) \in B_\varepsilon(\bar{z}, \bar{s})$, feasible to (P). Then, $s \in S \cap B_\varepsilon(\bar{s})$ and $z \in Z(s)$. It follows that $z \in Z(\bar{s})$. As $\bar{z} \in \arg\max\{c^T z : z \in Z(\bar{s})\}$, it must be $F(z, s) = c^T z \leq c^T \bar{z} = F(\bar{z}, \bar{s})$. Hence, $(\bar{z}, \bar{s})$ is a local solution of (P) and, by Theorem 3.2, an equilibrium point. ∎

It is known that the feasible set of LBP is nonconvex in general. However, by Proposition 3.1, it can be decomposed into a finite number of polyhedra, each one defined by $Z(s)$ for some $s \in S_v$. According to the above theorem, an equilibrium point $(\bar{z}, \bar{s}) \in Z_v \times S_v$ gives the best feasible vertex $\bar{z}$ in the face $Z(\bar{s})$ of Z.

In the rest of this section we derive sufficient conditions for local optimality in LBP to be satisfied by an equilibrium point.

Corollary 3.4 assures that a strict equilibrium point $(\bar{z}, \bar{s})$ gives a local optimum $\bar{z}$ of LBP. Actually, the strict equilibrium condition can be relaxed by the dual strict equilibrium condition. In this case, Theorem 3.4 guarantees that $\bar{z}$ is a local solution of LBP, provided that $S_v(\bar{z}) = \{\bar{s}\}$ by Corollary 3.3. This situation can be recognized by searching the adjcent vertices to $\bar{s}$, as shows the next corollary.

Corollary 4.1 *Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem $P(M)$. If $\bar{s} \in S_v$ and $\bar{z}^T s > 0$ for all $s \in S_v$ adjacent to $\bar{s}$, then $\bar{z}$ is a local solution of problem LBP.*

Proof: Let $(\bar{z}, \bar{s})$ be an equilibrium point with $\bar{s} \in S_v$. By Lemma 3.3, $\bar{s} \in S_v(\bar{z})$ and so $\bar{z}$ is feasible to LBP. In addition, Proposition 3.1 guarantees that $S(\bar{z})$ is a face of S. By hypothesis, $s \notin S(\bar{z})$ for all vertex $s \in S_v$ adjacent to $\bar{s}$. Hence, $S_v(\bar{z}) = \{\bar{s}\}$. By Theorem 3.4, $(\bar{z}, \bar{s})$ is a local solution of LBP. ∎

Another sufficient condition for local optimality in LBP is given below. It is weaker but more difficult to be verified computationally.

Theorem 4.2 *Let $(\bar{z}, \bar{s})$ be an equilibrium point of the penalized problem $P(M)$. If there is $\varepsilon > 0$ such that*

$$\bar{s} \in \bigcap_{z} \{S(z) : z \in Z \cap B_\varepsilon(\bar{z}), S(z) \neq \emptyset\}, \tag{9}$$

then $\bar{z}$ is a local solution of problem LBP. Particularly, if $\bar{s} \in S_v$ then $S(z)$ can be replaced by $S_v(z)$ in (9).

Proof: Let $(\bar{z}, \bar{s})$ be an equilibrium point. By Lemma 3.3, $\bar{s} \in S(\bar{z})$. Then, by Definition 3.1, $\max\{c^T z - \bar{M}\bar{s}^T z : z \in Z\} = c^T \bar{z}$ for some $\bar{M} \geq 0$. Let us recall that the feasible set of LBP is $\{z \in Z : S(z) \neq \emptyset\}$. Let us consider $\varepsilon > 0$ satisfying (9) and restrict LBP to $B_\varepsilon(\bar{z})$. It follows that

$$\begin{aligned} \max\{c^T z : z \in Z \cap B_\varepsilon(\bar{z}), S(z) \neq \emptyset\} &= \max\{c^T z - \bar{M}\bar{s}^T z : z \in Z \cap B_\varepsilon(\bar{z}), S(z) \neq \emptyset\} \\ &\leq \max\{c^T z - \bar{M}\bar{s}^T z : z \in Z\} = c^T \bar{z}. \end{aligned} \tag{10}$$

Since $\bar{z} \in Z \cap B_\varepsilon(\bar{z})$ and $S(\bar{z}) \neq \emptyset$, equality holds in (10). Hence, $\bar{z}$ is a local solution of LBP. Besides, as polyhedron $S(z)$ has no lines, $S_v(z) \neq \emptyset$ if $S(z) \neq \emptyset$. Then, we can replace $S(z)$ by $S_v(z)$ in (9) when $\bar{s} \in S_v$. ∎

Remark 4.1 Condition (9) is weaker than that one which is used by Corollary 4.1. In fact, that condition implies that $S_v(\bar{z}) = \{\bar{s}\}$. So, for $\varepsilon = \varepsilon(\bar{z})$ given by Lemma 3.1, condition (9) trivially

$$
\begin{array}{lll}
\text{(E2)} & \max & f_1(x,y) = -x \\
& \text{s.t.} & x \geq 0,\ y \text{ solves} \\
& & \quad \max \quad f_2(x,y) = -y \\
& & \quad \text{s.t.} \quad x + y \geq 2 \\
& & \qquad\qquad x - y \leq 0 \\
& & \qquad\qquad 2x + y \geq 2 \\
& & \qquad\qquad y \geq 0
\end{array}
$$

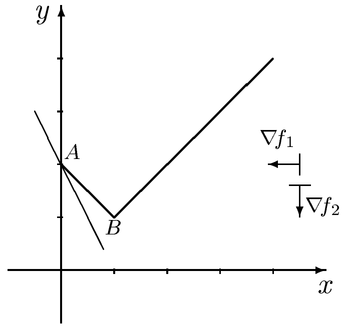

Figure 2:

holds. On the other hand, it may be the case that condition (9) does not imply $S_v(\bar{z}) = \{\bar{s}\}$. Indeed, let us consider example (E2), figure 2, where the feasible set is shown in bold. Let $\bar{z} = (\bar{x}, \bar{y}, \bar{w}_1, \bar{w}_2, \bar{w}_3)^T = (0,2,0,2,0)^T \in Z$, related to point A, and $\bar{s} = (0, \bar{v}, \bar{u}_1, \bar{u}_2, \bar{u}_3)^T = (0,0,1,0,0)^T \in S$. We have that $(\bar{z}, \bar{s})$ is an equilibrium point for $\bar{M} = 0$. In addition, let $\varepsilon \in (0,1)$ and $z = (x, y, w_1, w_2, w_3)^T \in B_\varepsilon(\bar{z})$, feasible to LBP. We can see that z lies on the segment (A, B). Thus, $w_1 = 0$ and so $\bar{s}^T z = 0$. Hence, $(\bar{z}, \bar{s})$ satisfies (9). However, the hipothesis of Corollary 4.1 is not verified because there is $\hat{s} \in S_v(\bar{z})$, adjacent to $\bar{s}$, namely $\hat{s} = (0,0,0,0,1)^T$.

Remark 4.2 Condition (9) is not necessary for an equilibrium point to be a local solution of LBP. Indeed, let us consider example (E2), figure 2, with the first level objective function replaced by $f_1(x,y) = -y$. Then, the global solution is attained at point B, which does verify condition (9).

Finnaly, we consider an assumption of nondegeneracy in order to get a more interesting characterization. We are going to use the following notation. Let be given $(\bar{z}, \bar{s}) \in Z_v \times S_v$, where $\bar{z}$ is a nondegenerate vertex defined by the basis E of $A = [E\ N]$. We denote by $\mathcal{N}(\bar{z})$ and $\mathcal{B}(\bar{s})$ the index sets of nonbasic variables at $\bar{z}$ and basic variables at $\bar{s}$, respectively. For $i \in \mathcal{N}(\bar{z})$, we have that $d_i^T = (d_{iE}^T, d_{iN}^T) = (-(E^{-1}N_i)^T, e_i^T)$ is an extreme direction of Z, where N_i and e_i respectively are the columns of matrices N and I_{n-m} corresponding to z_i.

Theorem 4.3 *Let $(\bar{z}, \bar{s}) \in Z_v \times S_v$ be an equilibrium point of the penalized problem where $\bar{z}$ is a nondegenerate vertex. Let $\mathcal{N}^+(\bar{z}) = \{i \in \mathcal{N}(\bar{z}) : c^T d_i > 0\}$. Then, $\mathcal{N}^+(\bar{z}) \subseteq \mathcal{B}(\bar{s})$. In addition, $\bar{z}$ is not a local solution of LBP if, and only if, there are $i \in \mathcal{N}^+(\bar{z})$ and $\hat{s} \in S_v(\bar{z})$ with $\hat{s}_i = 0$.*

Proof: Let $(\bar{z}, \bar{s}) \in Z_v \times S_v$ be an equilibrium point such that $\bar{z}$ is nondegenerate. First we prove that

$$s^T d_i = s_i \quad \forall i \in \mathcal{N}(\bar{z})\ \forall s \in S_v(\bar{z}). \tag{11}$$

In fact, let $i \in \mathcal{N}(\bar{z})$ and $s \in S_v(\bar{z})$. Since $s^T\bar{z} = 0$ and $\bar{z}_E > 0$, it must be $s_E = 0$. Hence, $s^T d_i = s_i - s_E^T E^{-1} N_i = s_i$.

To prove the desired inclusion, consider $i \in \mathcal{N}^+(\bar{z})$. By (11), it follows that $\bar{s}^T d_i = \bar{s}_i \geq 0$ because $\bar{s} \in S_v(\bar{z})$. Suppose that $\bar{s}^T d_i = 0$. As $\bar{z}$ is nondegenerate, there is $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z$. Then, $\bar{s}^T z = 0$ and $c^T z - c^T \bar{z} = \alpha c^T d_i > 0$. Thus, it results that $\bar{z} \notin \arg\max\{c^T z : z \in Z(\bar{s})\}$, which contradicts Theorem 4.1. Hence, $\bar{s}_i = \bar{s}^T d_i > 0$ and so $i \in \mathcal{B}(\bar{s})$. Therefore, $\mathcal{N}^+(\bar{z}) \subseteq \mathcal{B}(\bar{s})$.

Now we prove the claimed equivalence. Assume that $\bar{z}$ is not a local solution of LBP. Since $S_v(\bar{z}) \neq \emptyset$, by Theorem 3.4 there is $\hat{s} \in S_v(\bar{z})$ such that $(\bar{z}, \hat{s})$ is not an equilibrium point. By Theorem 4.1, $\bar{z} \notin \arg\max\{c^T z : z \in Z(\hat{s})\}$. Since $\bar{z} \in Z_v(\hat{s})$ and is nondegenerate, there must exist $i \in \mathcal{N}^+(\bar{z})$ and $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z(\hat{s})$. Therefore, $\hat{s}^T d_i = 0$ implying, by (11), that $\hat{s}_i = 0$. Conversely, let $i \in \mathcal{N}^+(\bar{z})$ and $\hat{s} \in S_v(\bar{z})$ with $\hat{s}_i = 0$. By (11), $\hat{s}^T d_i = \hat{s}_i = 0$. Let $\varepsilon > 0$ arbitrary. Since $\bar{z}$ is nondegenerate, there is $\alpha > 0$ such that $z = \bar{z} + \alpha d_i \in Z \cap B_\varepsilon(\bar{z})$. In addition, z is feasible to LBP because $\hat{s}^T z = 0$. Moreover, $c^T z - c^T \bar{z} = \alpha c^T d_i > 0$. Thus, $\bar{z}$ is not a local solution of LBP. ∎

5 A local algorithm

In this section we propose a local algorithm for LBP which essentially finds equilibrium points of $P(M)$. We assume that the vertices of Z are nondegenerate.

Algorithm 1

0. If $Z \times S = \emptyset$ then LBP is infeasible. Otherwise, let $z^0 \in Z$. Set $k = 1$.
1. Solve $\min\{(z^0)^T s : s \in S\}$ by the simplex method to obtain a solution $s^k \in S_v$.
2. Consider problem $P(s^k)$ defined as $\max\{c^T z : z \in Z, (s^k)^T z = 0\}$. Attempt to solve $P(s^k)$ by the simplex method. If $P(s^k)$ is unbounded, then LBP is unbounded. Otherwise, $P(s^k)$ has a solution $z^k \in Z_v$ and (z^k, s^k) is a local solution of (P).
3. If $\mathcal{N}^+(z^k) = \emptyset$, stop returning z^k as a global solution of LBP. If $\min\{s_i : s \in S, (z^k)^T s = 0\} > 0$ for each $i \in \mathcal{N}^+(z^k)$, stop returning z^k as a local solution of LBP. Otherwise, let $i \in \mathcal{N}^+(z^k)$ and $s^{k+1} \in S(z^k)$ with $s_i^{k+1} = 0$; set $k = k + 1$ and go to step 2.

We can show that problem $P(s^k)$ is equivalent to the parametric linear problem $\max\{c^T z - M(s^k)^T z : z \in Z\}$ for sufficiently large values of M. So, steps 1 and 2 of algorithm 1 can be viewed as one iteration of the mountain climbing algorithm (with the Big-M simplex method) applied to the parametric bilinear problem $P(M)$. The mountain climbing algorithm is used by Konno (Ref. 19) to find a KKT point of a bilinear problem. It alternatively solves the linear programs which are obtained when one of the variables is fixed until an "equilibrium" is attained. Such a procedure is also used by Audet *at al.* (Ref. 20) to calculate lower bounds in a branch-and-bound method.

In the case of the parametric problem $P(M)$, the equilibrium condition is achieved in the first iteration, i.e. by solving two linear programs. Moreover, the obtained point is a local solution of the MPEC (P). Let us note that it may not be as easy to find even stationary points of more general MPECs (Ref. 13).

The proof of the well-definition of algorithm 1 is given below. It uses the following preliminary result.

Lemma 5.1 *Problem $P(s^k)$ defined at step 2 is feasible.*

Proof: At step 2, we have that $Z \times S \neq \emptyset$. First assume that $k = 1$. Then $s^k \in S_v$ is a solution of the linear problem solved at step 1. Let $s^k = ((D^{-1}b)^T, 0)$, where D is a basis of $B = [D \;\; R]$. Let $(z^0)^T = (z_D^T, z_R^T)$. Consider the optimal reduced cost vetor $\tilde{z}^T = (0, z_R^T - z_D^T D^{-1} R) = z^T - z_D^T D^{-1} B \geq 0$ obtained at step 1. Since $AB^T = 0$, it follows that $A\tilde{z} = Az = a$. Hence, $\tilde{z} \in Z$. And the partitions of $\tilde{z}$ and s^k result that $\tilde{z}^T s^k = 0$. Now, assume that $k > 1$. Then, $s^k \in S(z^{k-1})$, according to step 3. In both cases, we have that $P(s^k)$ is feasible. ■

Proposition 5.1 *The algorithm is well-defined. It halts finding a local solution of problem LBP or verifying that LBP is infeasible or unbounded.*

Proof: Let us show that the algorithm is well-defined following its steps. The case of infeasibility is trivial. Otherwise, points z^0 and s^k can be obtained since $Z \times S \neq \emptyset$ and $(z^0)^T s \geq 0$ for all $s \in S$. By Lemma 5.1, $P(s^k)$ is feasible. Since $P(s^k)$ is more restricted than (P), which is globally equivalent to LBP, if $P(s^k)$ is unbounded so is LBP. If $P(s^k)$ is not unbounded, then point z^k is well-defined. By Theorem 4.1, (z^k, s^k) is an equilibrium point and so, by Theorem 3.2, it is a local solution of (P). The conclusions of local optimality at step 3 are due to Theorem 4.3. Finally, we have that $c^T z^{k+1} > c^T z^k$, since there is $z \in Z(s^{k+1})$ such that $c^T z > c^T z^k$ (see proof of Theorem 4.3). Therefore, the algorithm halts with one of the three possibilities. ■

It is worth saying that a good initial point z^0 is a solution of the problem $\max\{c^T z : z \in Z\}$, if it is not unbounded. Let us note that this problem is a relaxation of LBP, usually called the leader's relaxation. Many algorithms proposed in the literature start at this point (Refs. 3,7,8). Observe that if $\mathcal{N}^+(z^k) = \emptyset$ then z^k is a solution of this relaxation and so a global solution of LBP.

In order to check whether $P(s^k)$ is unbounded or to find a solution z^k and simultaneously determine $\mathcal{N}^+(z^k)$, we can solve the problem $P'(s^k) \equiv \min\{(s^k)^T z : z \in Z\}$, whose optimal

(E3) $\max f_1(x,y) = -x + 2y_1 - 8y_2$
s.t. $x \geq 0,\ y = (y_1, y_2)$ solves
$\max\ f_2(x,y) = y_1 + 2y_2$
s.t. $x + y_1 + y_2 \leq 3$
$x + y_1 - y_2 \geq 1$
$-x + y_1 + y_2 \leq 1$
$x - y_1 + y_2 \leq 1$
$x - y_1 + 5y_2 \leq 2$
$y_1 \geq 0,\ y_2 \geq 0$

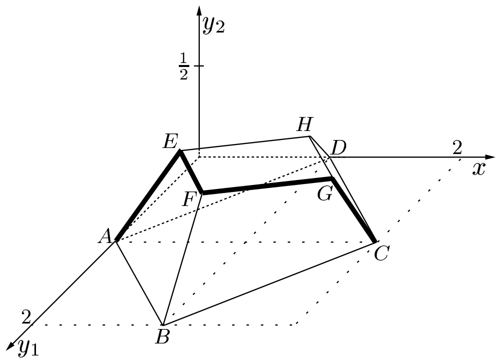

Figure 3:

value is zero, and then maximize $c^T z$ over the optimal set of $P'(s^k)$. This can be done from the optimal tableau of $P'(s^k)$ by allowing to enter the basis only the variables whose reduced cost related to s^k is null. This way the reduced costs related do c give the values $c^T d_i$ for $i \in \mathcal{N}(z^k)$.

At step 3, before solving the problems $\min\{s_i : s \in S, (z^k)^T s = 0\}$, it may be useful to solve $\min\{(z^k)^T s : s \in S\}$. If $k = 1$, this can be readly done from the optimal tableau obtained at step 1. Note that s^k is a solution to this problem, since $(s^k)^T z^k = 0$. If s^k is the unique extreme solution, the algorithm can stop returning z^k as a local solution, according to Corollary 4.1. Otherwise, each problem $\min\{s_i : s \in S, (z^k)^T s = 0\}$ can be solved from the optimal tableau of this preliminary problem. Let us recall that $i \in \mathcal{N}^+(z^k) \subseteq \mathcal{B}(s^k)$. So, the goal is to take s_i out of the basis while keeping $(z^k)^T s = 0$.

5.1 Numerical Example

In order to illustrate how the algorithm works we consider the example (E3), figure 3, which was adapted from Ref. 2. The feasible set is the union of the edges AE, EF, FG and GC. Point A is the global solution and point C is a local solution. We have that

$$\begin{aligned} Z &= \{z = (x, y_1, y_2, w_1, w_2, w_3, w_4, w_5)^T \geq 0 :\ x + y_1 + y_2 + w_1 = 3,\ x + y_1 - y_2 - w_2 = 1, \\ &\quad -x + y_1 + y_2 + w_3 = 1,\ x - y_1 + y_2 + w_4 = 1,\ x - y_1 + 5y_2 + w_5 = 2\}, \\ S &= \{s = (0, v_1, v_2, u_1, u_2, u_3, u_4, u_5)^T \geq 0 :\ u_1 - u_2 + u_3 - u_4 - u_5 - v_1 = 1, \\ &\quad u_1 + u_2 + u_3 + u_4 + 5u_5 - v_2 = 2\}. \end{aligned}$$

Let us start the algorithm at the solution of the leader's relaxation $z^0 = (1,2,0,0,2,0,2,3)^T$, which corresponds to point B. A solution obtained by the simplex method at step 2 is $s^1 = (0,0,0,7/6,0,0,0,1/6)^T$. The solution of problem $P(s^1)$ is $z^1 = (7/4,1,1/4,0,3/2,3/2,0,0)^T$, which is related to point G. We have that (z^1, s^1) is an equilibrium point or, equivalently, a local solution of (P). Observe that z^1 is the best point in the edge FG which is the face $Z(s^1)$. At step 3, we have that $\mathcal{N}^+(z^1) = \{8\}$. The solution of the minimization problem then considered is $s^2 = (0,0,0,3/2,0,0,1/2,0)^T$ with optimal value $s_8^2 = 0$. This means that z^1 is not a local solution of (E3).

We return to step 2 with $k = 2$. The solution of problem $P(s^2)$ is $z^2 = (2,1,0,0,2,2,0,1)^T$ corresponding to point C. Then, (z^2, s^2) is a new equilibrium point that is the best feasible point in the face GC. Now, $\mathcal{N}^+(z^2) = \{7\}$ and the optimal value of the problem considered at step 3 is positive. Thus, the algorithm stops returning z^2 as a local solution. As remarked this conclusion could have been infered by regarding that the problem $\min\{(z^2)^T s : s \in S\}$ has a unique extreme solution s^2 (although its optimal set is not a singleton).

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