

# The Interleaved Multichromatic Number of a Graph

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## Abstract

For  $k \geq 1$ , we consider interleaved  $k$ -tuple colorings of the nodes of a graph, that is, the assignment of  $k$  distinct natural numbers to each node in such a way that nodes that are connected by an edge receive numbers that are strictly alternating between them with respect to the relation  $<$ . If it takes at least  $\chi_{\text{int}}^k(G)$  distinct numbers to provide graph  $G$  with such a coloring, then the interleaved multichromatic number of  $G$  is  $\chi_{\text{int}}^*(G) = \inf_{k \geq 1} \chi_{\text{int}}^k(G)/k$  and is known to be given by a function of the simple cycles of  $G$  under acyclic orientations if  $G$  is connected. This paper contains a new proof of this result. Unlike the original proof, the new proof makes no assumptions on the connectedness of  $G$ , nor does it resort to the possible applications of interleaved  $k$ -tuple colorings and their properties.

**Keywords:** Chromatic number, multichromatic number, interleaved multichromatic number, acyclic orientations.

## 1. Introduction

Let  $G = (N, E)$  be an undirected graph with  $n = |N|$  and  $m = |E| \geq 1$ . For  $k \geq 1$ , a  $k$ -tuple coloring of  $G$  is an assignment of  $k$  distinct natural numbers (*colors*) to each of the nodes of  $G$  in such a way that no two nodes connected by an edge are assigned an identical color. The least number of colors with which  $G$  can be  $k$ -tuple colored is its  $k$ -chromatic number, denoted by  $\chi^k(G)$ . For  $k = 1$ , the 1-chromatic number of  $G$  is the graph's *chromatic number*, denoted by  $\chi(G)$ .

There is also a notion of “efficiency” that goes along  $k$ -tuple colorings of  $G$ , which is the notion that, for two distinct positive integers  $k_1$  and  $k_2$ , it is more “efficient” to  $k_1$ -tuple color  $G$  than to  $k_2$ -tuple color it if and only if  $\chi^{k_1}(G)/k_1 < \chi^{k_2}(G)/k_2$ . This notion is formalized by the definition of the *multichromatic number* of  $G$  [5], denoted by  $\chi^*(G)$  and given by

$$\chi^*(G) = \inf_{k \geq 1} \frac{\chi^k(G)}{k}. \quad (1)$$

Because the infimum in (1) can be shown to be always attained [2], the  $k$  for which the ratio in (1) is minimum gives the most “efficient”  $k$ -tuple coloring of  $G$ .

One special case of  $k$ -tuple colorings that has great practical interest is the case of *interleaved  $k$ -tuple colorings*, defined as follows. For  $i \in N$ , let  $c_i^1, \dots, c_i^k$  be the  $k$  colors assigned to node  $i$ . This  $k$ -tuple coloring is interleaved if and only if either  $c_i^1 < c_j^1 < \dots < c_i^k < c_j^k$  or  $c_j^1 < c_i^1 < \dots < c_j^k < c_i^k$  for all  $(i, j) \in E$ . Every 1-tuple coloring is interleaved, and the counterparts for interleaved colorings of the graph’s  $k$ -chromatic and multichromatic numbers are, respectively, its *interleaved  $k$ -chromatic number* and *interleaved multichromatic number*. These are denoted, respectively, by  $\chi_{\text{int}}^k(G)$  and  $\chi_{\text{int}}^*(G)$ . Following (1), the latter is defined as

$$\chi_{\text{int}}^*(G) = \inf_{k \geq 1} \frac{\chi_{\text{int}}^k(G)}{k}. \quad (2)$$

Clearly,  $\chi^*(G) \leq \chi_{\text{int}}^*(G) \leq \chi(G)$ .

For connected  $G$ , the infimum in (2) has also been shown to be always attained and given as follows [1]. Let  $K$  be the set of all simple cycles of  $G$ . For  $\kappa \in K$ , let  $\kappa^+$  and  $\kappa^-$  be the two possible traversal directions of  $\kappa$ . If  $\omega$  is an *acyclic orientation* of  $G$  (that is, an assignment of directions to edges that forms no directed cycles), then we denote by  $m(\kappa^+, \omega)$  the number of edges in  $\kappa$  oriented by  $\omega$  in the  $\kappa^+$  direction. Likewise for  $m(\kappa^-, \omega)$ . What has been shown for nonempty  $K$  is that

$$\frac{1}{\chi_{\text{int}}^*(G)} = \max_{\omega \in \Omega(G)} \min_{\kappa \in K} \frac{\min\{m(\kappa^+, \omega), m(\kappa^-, \omega)\}}{|\kappa|}, \quad (3)$$

where  $\Omega(G)$  is the set of all acyclic orientations of  $G$  and  $|\kappa|$  is the size of  $\kappa$ . If  $K = \emptyset$  (i.e.,  $G$  is a tree), then  $\chi_{\text{int}}^*(G) = 2$ .

In [1],  $1/\chi_{\text{int}}^*(G)$  is analyzed as a measure of concurrency (or parallelism) for certain distributed computations over systems that can be represented by connected graphs, and (3) is obtained through proofs that make heavy use of both the graph’s connectedness and properties of those distributed computations. What we do in this paper is to provide an alternative proof for (3), one that does not make assumptions on the connectedness of  $G$  and is built from “first principles,” that is, independently of the possible applications of the concept of “efficient” interleaved  $k$ -tuple colorings. This is done in Section 2, and in Section 3 we give concluding remarks and argue that (3) provides both “efficient” interleaved  $k$ -tuple colorings and optimal 1-tuple colorings with a common underlying relationship to the set  $\Omega(G)$  of the graph’s acyclic orientations.

## 2. A new proof

Our proof is based on the *lexicographic product* [4] of  $G$  and the complete graph on  $k$  nodes. This lexicographic product is another undirected graph, denoted by  $G^k$  and having  $k$  nodes for each node in  $G$ . For  $i \in N$ , we denote the nodes of  $G^k$  by  $i^1, \dots, i^k$ . The edges of  $G^k$  are deployed in such a way that, for fixed  $i$ , an edge exists connecting every pair of nonidentical nodes in  $i^1, \dots, i^k$ , and for  $(i, j) \in E$ , an edge exists between each one of  $i^1, \dots, i^k$  and each one of  $j^1, \dots, j^k$ . Clearly, every  $k$ -tuple coloring of  $G$  corresponds to a 1-tuple coloring of  $G^k$ , and conversely.

Now consider the set  $\Omega(G^k)$  of the acyclic orientations of  $G^k$ . We say that  $\varphi \in \Omega(G^k)$  is a *layered* orientation of  $G^k$  if and only if every edge  $(i^{k_1}, j^{k_2})$  for which  $k_1 > k_2$  is oriented from  $i^{k_1}$  to  $j^{k_2}$ , and furthermore every edge  $(i^{k_1}, j^{k_1})$  gets the same orientation for  $k_1 = 1, \dots, k$ . If we visualize  $G^k$  as  $k$  instances of  $G$  stacked upon one another in such a way that the instance at the bottom contains superscript-1 nodes, the one right above it superscript-2 nodes, and so on, then what a layered acyclic orientation of  $G^k$  does is to provide the  $k$  instances of  $G$  with identical acyclic orientations from  $\Omega(G)$  and orient all other edges (the ones that connect nodes from different instances of  $G$ ) from top to bottom. The set of all layered acyclic orientations of  $G^k$  is denoted by  $\Omega^L(G^k)$ .

In  $G^k$ , we refer to each of the aforementioned instances of  $G$  as a *layer*, and number each layer with the superscript of the nodes that it contains. Given a layered acyclic orientation, every maximal directed path starts in layer  $k$  (because that is where all *sources* are, these being nodes whose adjacent edges are all oriented outward) and ends in layer 1 (which contains all *sinks*, that is, nodes whose adjacent edges are all oriented inward).

For a generic undirected graph  $H$  and for an acyclic orientation  $\omega$  of  $H$ , let  $l_\omega$  be the number of nodes in the longest directed path in  $H$  according to  $\omega$ . Also, let a 1-tuple coloring of  $H$  be called *monotonic* with respect to an acyclic orientation  $\omega$  of  $H$  if and only if the colors it assigns decrease along any directed path according to  $\omega$ . If  $\Omega(H)$  is the set of acyclic orientations of  $H$ , then we have the following, for whose proof the reader is referred to [3].

**Lemma 1.** *To every  $\omega \in \Omega(H)$  there corresponds a monotonic 1-tuple coloring of  $H$  with  $l_\omega$  colors. Conversely, to every 1-tuple coloring of  $H$  by a total of  $c$  colors there corresponds an orientation  $\omega \in \Omega(H)$  with respect to which the coloring is monotonic and such that  $l_\omega \leq c$ .*

By Lemma 1, the chromatic number of graph  $H$  is given by the number of nodes on the longest directed path that is shortest among all possible acyclic orientations of  $H$  [3]. In the case of  $G^k$ , and considering that  $\chi^k(G) = \chi(G^k)$ , we have  $\chi^k(G) = \min_{\varphi \in \Omega(G^k)} l_\varphi$ . Lemma 1 is the basis of our first supporting result, which states that it suffices to look at the orientations in  $\Omega^L(G^k) \subseteq \Omega(G^k)$  to obtain  $\chi_{\text{int}}^k(G)$ .

**Lemma 2.** *For  $k \geq 1$ ,  $\chi_{\text{int}}^k(G) = \min_{\varphi \in \Omega^L(G^k)} l_\varphi$ .*

**Proof:** Let  $\varphi$  be a layered acyclic orientation of  $G^k$ . By Lemma 1,  $G^k$  can be colored monotonically by a 1-tuple coloring that employs a total of  $l_\varphi$  colors. If  $(i, j) \in E$  is oriented by  $\varphi$  within each layer from the corresponding instance of  $i$  to the corresponding instance of  $j$ , then by definition of a layered orientation a directed path exists from  $i^k$  to  $j^1$  that alternates instances of  $i$  and  $j$  along its way (that is, a path like  $i^k \rightarrow j^k \rightarrow i^{k-1} \rightarrow j^{k-1} \rightarrow \dots \rightarrow i^1 \rightarrow j^1$ ). It then follows from the monotonicity of the 1-tuple coloring of  $G^k$  that such a coloring corresponds to an interleaved  $k$ -tuple coloring of  $G$ , thus

$$\chi_{\text{int}}^k(G) \leq \min_{\varphi \in \Omega^L(G^k)} l_\varphi.$$

Conversely, consider an interleaved  $k$ -tuple coloring of  $G$  by a total of  $c$  colors. This coloring corresponds to a 1-tuple coloring of  $G^k$  by the same  $c$  colors. By Lemma 1, an orientation  $\varphi$  of  $G^k$

exists with respect to which this 1-tuple coloring of  $G^k$  is monotonic and as such orients all edges of  $G^k$  from the node with higher color to the one with lower. For an arbitrary edge  $(i, j)$  of  $G$ , this, together with the fact that the assumed  $k$ -tuple coloring of  $G$  is interleaved, implies that  $\varphi$  is layered. In addition, Lemma 1 also ensures that  $l_\varphi \leq c$ , hence

$$\chi_{\text{int}}^k(G) \geq \min_{\varphi \in \Omega^L(G^k)} l_\varphi. \quad \blacksquare$$

Our next supporting result is a statement on the morphology of longest directed paths according to layered acyclic orientations of  $G^k$ . What it states is that such a directed path's every edge is either confined to a layer or joins two adjacent layers. In the latter case, it also states that, if such an edge exists from  $i^{k_1}$  to  $j^{k_1-1}$ , then a directed edge also exists from  $j^{k_1}$  to  $i^{k_1}$ .

**Lemma 3.** *For  $k \geq k_1 \geq k_2 \geq 1$  and  $\varphi \in \Omega^L(G^k)$ , let  $p$  be a longest directed path according to  $\varphi$ , and let  $i^{k_1}$  and  $j^{k_2}$  be such that a directed edge exists on  $p$  from  $i^{k_1}$  to  $j^{k_2}$ . Then, for  $\ell = 1, \dots, k$ , either  $k_1 = k_2$  and a directed edge exists from  $i^\ell$  to  $j^\ell$ , or  $k_1 = k_2 + 1$  and a directed edge exists from  $j^\ell$  to  $i^\ell$ .*

**Proof:** What needs to be shown is that two specific scenarios cannot happen if  $\varphi$  is layered. First, suppose that  $k_1 = k_2 + 1$  and  $i = j$ , that is,  $p$  contains an edge directed from  $i^{k_1}$  to  $i^{k_1-1}$ . That this cannot happen follows from the fact that replacing this edge on  $p$  with either the directed path  $i^{k_1} \rightarrow r^{k_1} \rightarrow i^{k_1-1}$  or the directed path  $i^{k_1} \rightarrow r^{k_1-1} \rightarrow i^{k_1-1}$  for some  $(i, r) \in E$  yields a directed path longer than  $p$ . Because  $m \geq 1$ , at least one of the two possibilities is certain to occur, so we have a contradiction.

The second scenario corresponds to  $k_1 = k_2 + 1$  and  $(i, j) \in E$  when every layer  $\ell$  contains an edge directed from  $i^\ell$  to  $j^\ell$ . In this case, an edge exists directed from  $i^{k_1}$  to  $j^{k_1-1}$  on  $p$ . Once again, this edge can be replaced with the directed path  $i^{k_1} \rightarrow j^{k_1} \rightarrow i^{k_1-1} \rightarrow j^{k_1-1}$  to yield a directed path longer than  $p$ , also a contradiction.  $\blacksquare$

By Lemma 3, a longest directed path  $p$  in  $G^k$  according to a layered acyclic orientation  $\varphi$  looks like the following. It starts at a source in layer  $k$ , winds its way through the  $k$  layers, and ends at a sink in layer 1. In addition, each of the  $k - 1$  edges that lead from one layer to the next, say for example an edge from  $i^\ell$  to  $j^{\ell-1}$  for  $1 < \ell \leq k$ , is such that an edge exists in every layer  $\ell$  directed from  $j^\ell$  to  $i^\ell$ . If we let  $\omega \in \Omega(G)$  be the acyclic orientation of  $G$  that corresponds to the orientation of a layer by  $\varphi$  (the same for all layers), then  $p$  can be seen to leave a *trace* on  $G$  oriented by  $\omega$  as well. This trace is an undirected path in  $G$  that can be traversed by following  $\omega$  on the edges that correspond to  $p$  being confined to a layer, and by going against  $\omega$  on the  $k - 1$  edges that correspond to the transition of  $p$  from one layer to the next.

Next we give our main result, in which (3) is established.

**Theorem 4.** *If  $G$  is a forest, then  $\chi_{\text{int}}^*(G) = 2$ . Otherwise, then*

$$\chi_{\text{int}}^*(G) = \min_{\omega \in \Omega(G)} \max_{\kappa \in \mathbb{K}} \frac{|\kappa|}{\min\{m(\kappa^+, \omega), m(\kappa^-, \omega)\}}.$$

**Proof:** By definition of  $\chi_{\text{int}}^*(G)$  (cf. (2)) and by Lemma 2, we have

$$\begin{aligned}\chi_{\text{int}}^*(G) &= \inf_{k \geq 1} \frac{\chi_{\text{int}}^k(G)}{k} \\ &= \inf_{k \geq 1} \frac{1}{k} \min_{\varphi \in \Omega^L(G^k)} l_{\varphi}.\end{aligned}$$

Also, the definition of layered acyclic orientations of  $G^k$  implies that there exists a one-to-one correspondence between such orientations and the acyclic orientations of  $G$ . If, for  $k \geq 1$  and  $\omega \in \Omega(G)$ , we let  $\varphi_{\omega}^k$  be the corresponding orientation in  $\Omega^L(G^k)$ , then we can further write

$$\begin{aligned}\chi_{\text{int}}^*(G) &= \inf_{k \geq 1} \frac{1}{k} \min_{\omega \in \Omega(G)} l_{\varphi_{\omega}^k} \\ &= \min_{\omega \in \Omega(G)} \inf_{k \geq 1} \frac{l_{\varphi_{\omega}^k}}{k}.\end{aligned}$$

If  $G$  is a forest, then the theorem follows trivially. If not, then it remains to argue that

$$\max_{\kappa \in \mathbb{K}} \frac{|\kappa|}{\min\{m(\kappa^+, \omega), m(\kappa^-, \omega)\}} = \inf_{k \geq 1} \frac{l_{\varphi_{\omega}^k}}{k}$$

for all  $\omega \in \Omega(G)$ .

Let  $\hat{\kappa}$  be the simple cycle of  $G$  for which  $|\kappa|/\min\{m(\kappa^+, \omega), m(\kappa^-, \omega)\}$  is maximum over  $\mathbb{K}$ . Let also  $f = m(\hat{\kappa}^+, \omega)$  and  $b = m(\hat{\kappa}^-, \omega)$ , and assume that  $f \geq b$ . For  $k \geq 1$ , let  $p_k$  denote the longest directed path on  $G^k$  oriented by  $\varphi_{\omega}^k$ .

Because  $p_k$  is a longest directed path, then  $l_{\varphi_{\omega}^k}/(k-1)$  is maximum over all directed paths of  $G^k$  oriented by  $\varphi_{\omega}^k$ . In particular, if  $k-1$  is a multiple of  $b$ , in which case the trace implied by Lemma 3 of  $p_k$  on  $G$  oriented by  $\omega$  goes against  $\omega$  on exactly  $tb$  edges for some  $t \geq 1$ , then such a trace must coincide, except possibly for the first and last segments on which it agrees with  $\omega$ , with  $\hat{\kappa}$ . This is so because, by assumption,  $|\hat{\kappa}|$  is the largest possible contribution to a directed path in  $G^k$  if the path's trace on  $G$  oriented by  $\omega$  has to go against  $\omega$  on exactly  $b$  edges. If we let  $f_0$  and  $f_{\text{f}}$  be the numbers of edges in the first and last segments on which the trace of  $p_k$  and  $\omega$  agree, then

$$\begin{aligned}\frac{l_{\varphi_{\omega}^k}}{k} &= \frac{1 + f_0 + tf + tb + f_{\text{f}}}{1 + tb} \\ &= 1 + \frac{f_0 + tf + f_{\text{f}}}{1 + tb}.\end{aligned}$$

Now, the fact that  $p_k$  is longest also implies that at least one of the first and last segments on which  $p_k$  and  $\omega$  agree should encompass at least as many edges as  $\hat{\kappa}$ 's largest segment in the  $\hat{\kappa}^+$  direction according to  $\omega$ . If such a segment contains  $f_+$  edges, then this means that  $f_0 + f_{\text{f}} \geq f_+ \geq f/b$ , which yields

$$\begin{aligned}\frac{l_{\varphi_{\omega}^k}}{k} &\geq 1 + \frac{f}{b} \\ &= \frac{f + b}{b} \\ &= \frac{|\hat{\kappa}|}{m(\hat{\kappa}^-, \omega)}.\end{aligned}$$

Furthermore, we also have

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{l_{\varphi_{\omega}^k}}{k} &= \frac{f + b}{b} \\ &= \frac{|\hat{\kappa}|}{m(\hat{\kappa}^-, \omega)}. \end{aligned}$$

Finally, we must consider the cases in which  $k - 1$  is not a multiple of  $b$ . In these cases, it may happen that the trace of  $p_k$  does not coincide with  $\hat{\kappa}$  on the bottommost layer at all, which merely indicates that, on that layer, alternatives exist that lead to a directed path longer than the one that is confined to  $\hat{\kappa}$ . This means that in such cases, too,

$$\frac{l_{\varphi_{\omega}^k}}{k} \geq \frac{|\hat{\kappa}|}{m(\hat{\kappa}^-, \omega)}.$$

We then have

$$\frac{|\hat{\kappa}|}{m(\hat{\kappa}^-, \omega)} = \inf_{k \geq 1} \frac{l_{\varphi_{\omega}^k}}{k},$$

which concludes the proof. ■

### 3. Discussion

This paper contains a new proof of (3). This new proof, in contrast to the original proof that appears in [1], does not make assumptions on the connectedness of  $G$ , nor does it resort to the specifics of possible applications of the concept of interleaved  $k$ -tuple colorings. Unlike the proof of [1], however, the proof in this paper does not establish that the infimum in (2) is always achieved, which is indeed the case. For this, the original proof continues to be the source, as establishing that property is achieved independently of the connectedness of  $G$ .

As expressed in Theorem 4,  $\chi_{\text{int}}^*(G)$  corresponds to the minimum, taken over all of the acyclic orientations of  $G$ , of a function of certain undirected paths of  $G$  under those acyclic orientations. Specifically, if for  $\omega \in \Omega(G)$  we let  $P(\omega)$  comprise every path  $p$  in  $G$  such that  $p$  contains all nodes in a simple cycle and starts at a node that is a source according to  $\omega$  if only the edges on the simple cycle are considered, then  $\chi_{\text{int}}^*(G)$  can be written as

$$\chi_{\text{int}}^*(G) = \min_{\omega \in \Omega(G)} \max_{p \in P(\omega)} \frac{l(p)}{1 + c(p, \omega)}, \quad (4)$$

where  $l(p)$  is the number of nodes of  $p$  and  $c(p, \omega)$  is the number of edges on  $p$  that are oriented by  $\omega$  contrary to the traversal of  $p$  from its starting node.

Interestingly, the expression in (4) is the same as the aforementioned consequence of Lemma 1 that

$$\chi(G) = \min_{\omega \in \Omega(G)} \max_{p \in P(\omega)} l(p), \quad (5)$$

if only we now let  $P(\omega)$  stand for the set of directed paths of  $G$  according to  $\omega$ , since in this case  $c(p, \omega) = 0$  for all  $\omega \in \Omega(G)$  and all  $p \in P(\omega)$ .

Since by (4) and (5), respectively, it is possible to assess a graph's interleaved multichromatic number and chromatic number by looking at functions of the graph's paths as oriented by the acyclic orientations of  $G$ , one interesting question is whether  $\chi^*(G)$ , the multichromatic number of  $G$ , is also amenable to a similar characterization. Unlike the other two cases, however, there does not appear to be a result like Lemma 2, which establishes the fundamental one-to-one correspondence between the acyclic orientations of  $G^k$  that matter and those of  $G$ . Finding such a characterization for  $\chi^*(G)$  remains then an open question.

### Acknowledgments

The author acknowledges partial support from CNPq, CAPES, the PRONEX initiative of Brazil's MCT under contract 41.96.0857.00, and a FAPERJ BBP grant.

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