Competition in randomly growing processes

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Based on joint works with

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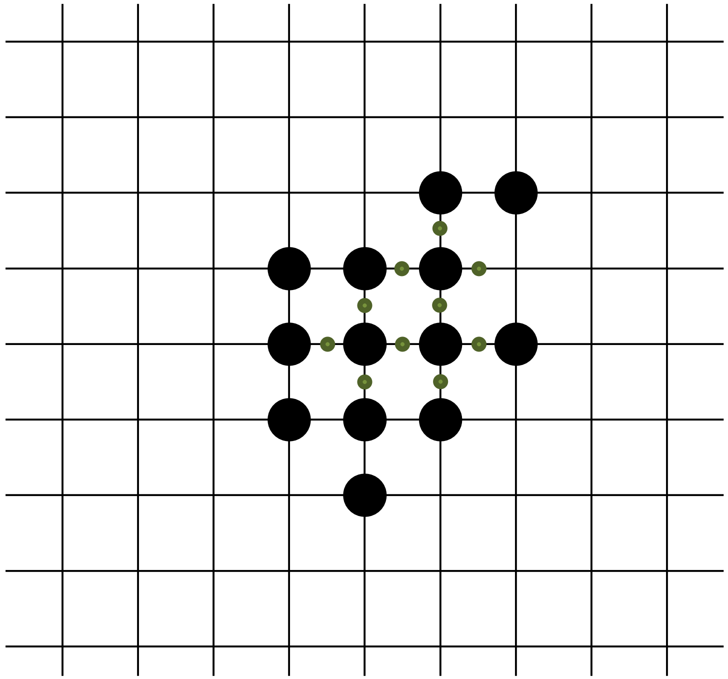
First Passage Percolation (FPP)



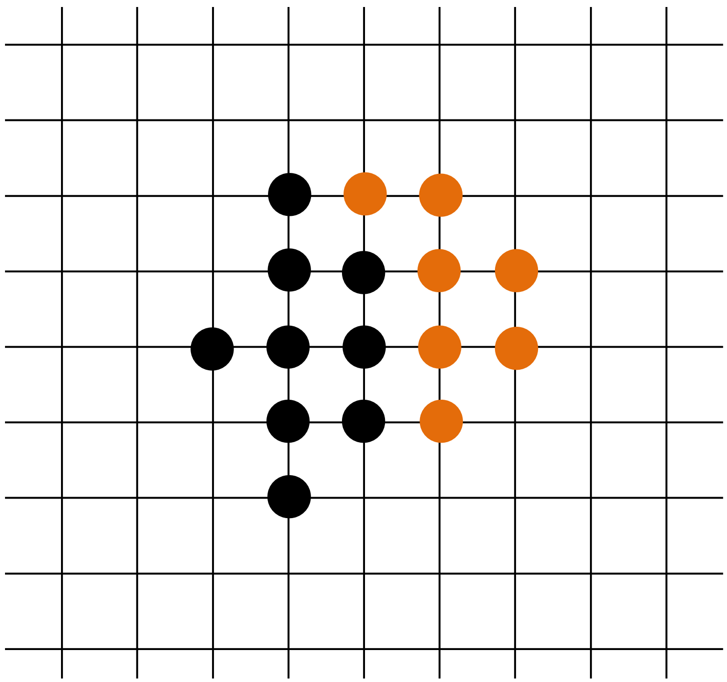
Shape theorem: forms a ball

- Start from the origin of $\mathbb{Z}^d$
- Grow by adding boundary edge at rate 1
  - A boundary edge is added u.a.r.
  - Defines a random metric: each edge has random weight $\sim$Exp(1)

First passage percolation (FPP):

- Eden (1961) to model problems in cell reproduction
- Hammersley and Welsh (1965) for general graphs and general passage times

FPP Competition



Two-type Richardson Model

Start from neighboring vertices

- Type 1 performs FPP at rate 1
- Type 2 performs FPP at rate λ

Each vertex gets occupied by the type that arrives to it first

Main questions

1. Which type produces an infinite cluster? (survival)
2. Is there coexistence? (i.e., both types produce infinite clusters)

Two-type Richardson Model

Theorem 1 [coexistence]

$\mathbb{P}(\text{coexistence}) > 0$ if $\lambda = 1$

O. Haggstrom and R. Pemantle. First passage percolation and a model for competing spatial growth. *Journal of Applied Probability*, 1998

C. Hoffman. Coexistence for Richardson type competing spatial growth models. *Annals of Applied Probability*, 2005

O. Garet and R. Marchand. Coexistence in two-type first-passage percolation models. *Annals of Applied Probability*, 2005

Conjecture

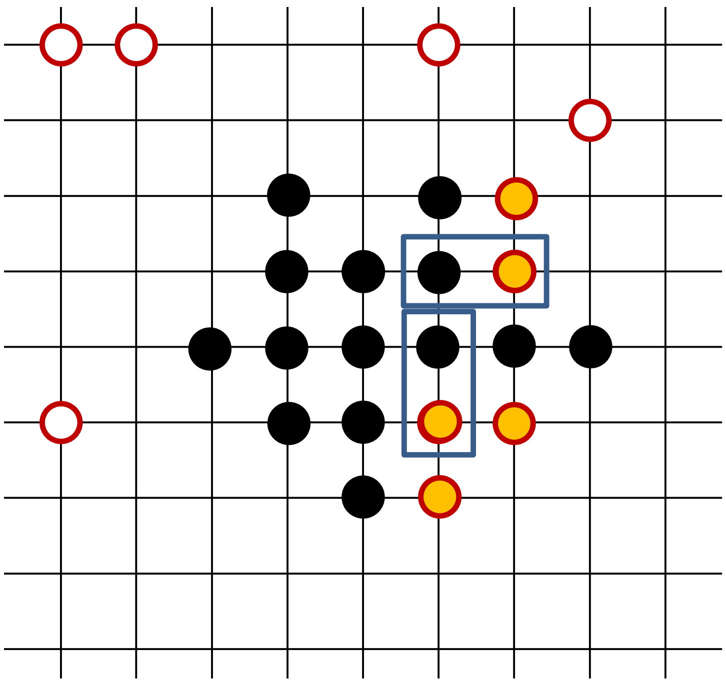
$\mathbb{P}(\text{coexistence}) > 0$ iff $\lambda = 1$

Theorem 2 [no coexistence]

$\mathbb{P}(\text{coexistence}) = 0$ for all but countably many values of λ

O. Haggstrom and R. Pemantle. Absence of mutual unbounded growth for almost all parameter values in the two-type Richardson model. *Stochastic Processes and their applications*, 2000

FPP in hostile environment



Type 1 starts from the origin
- Perform FPP at rate 1

Type 2 starts from seeds of IID Bern(p), which
- Do not evolve from time 0
- get *activated* when type 1 tries to occupy it
- After activation, evolve as FPP at rate λ

Main questions

- Which type produces an infinite cluster?
  (Type 2 is always an infinite set)
- Is there coexistence?

No monotonicity!
*Adding **Type 2 seeds** may speed up **Type 1**.*

Focus on case $p < 1 - p_c^{\text{site}}$ (i.e., $1 - p > p_c^{\text{site}}$)
so $\mathbb{Z}^d \setminus \{\text{seeds}\}$ has an infinite cluster

Motivation

Study of dendritic formation

- Invented as a tool to analyze a model from dendritic growth

Bacteria under starvation

Crystal dendrite

Dielectric breakdown

Spread of fake news

- Type 1 represents spread of fake news
- Type 2 spreads the correct information

First Result

Theorem [Survival of type 1 for small p]

For any $\lambda < 1$, there exists $p_0 \in (0,1)$ such that $\forall p < p_0$

1. $\mathbb{P}(\textbf{Type 1 } \text{survives}) > 0$
2. $\mathbb{P}(\forall t \geq 0,\ \overline{\text{Type1}_t} \supset \text{Ball}(ct)) > 0,$

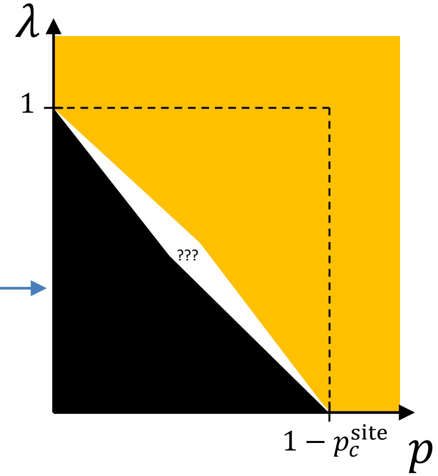
   where $\overline{\text{Type1}_t} = \text{Type1}_t \cup \text{"finite components of Type1}^c\text{"}$

λ = rate of type 2
p = density of type 2 seeds

V. Sidoravicius and A. S. Multi-particle diffusion limited aggregation. *Inventiones Mathematicae*, to appear

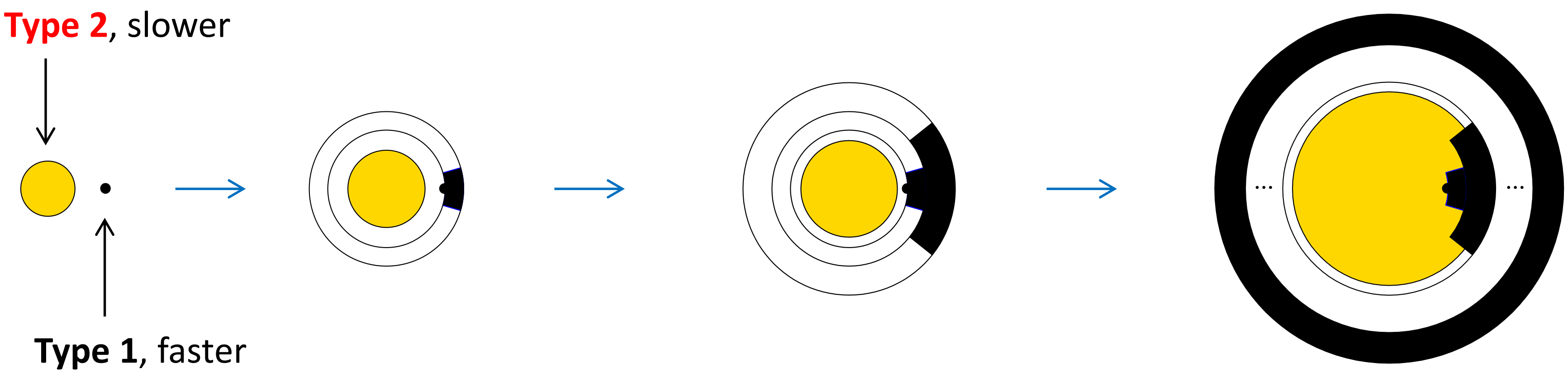
Expected behavior:

Sidoravicius, S.



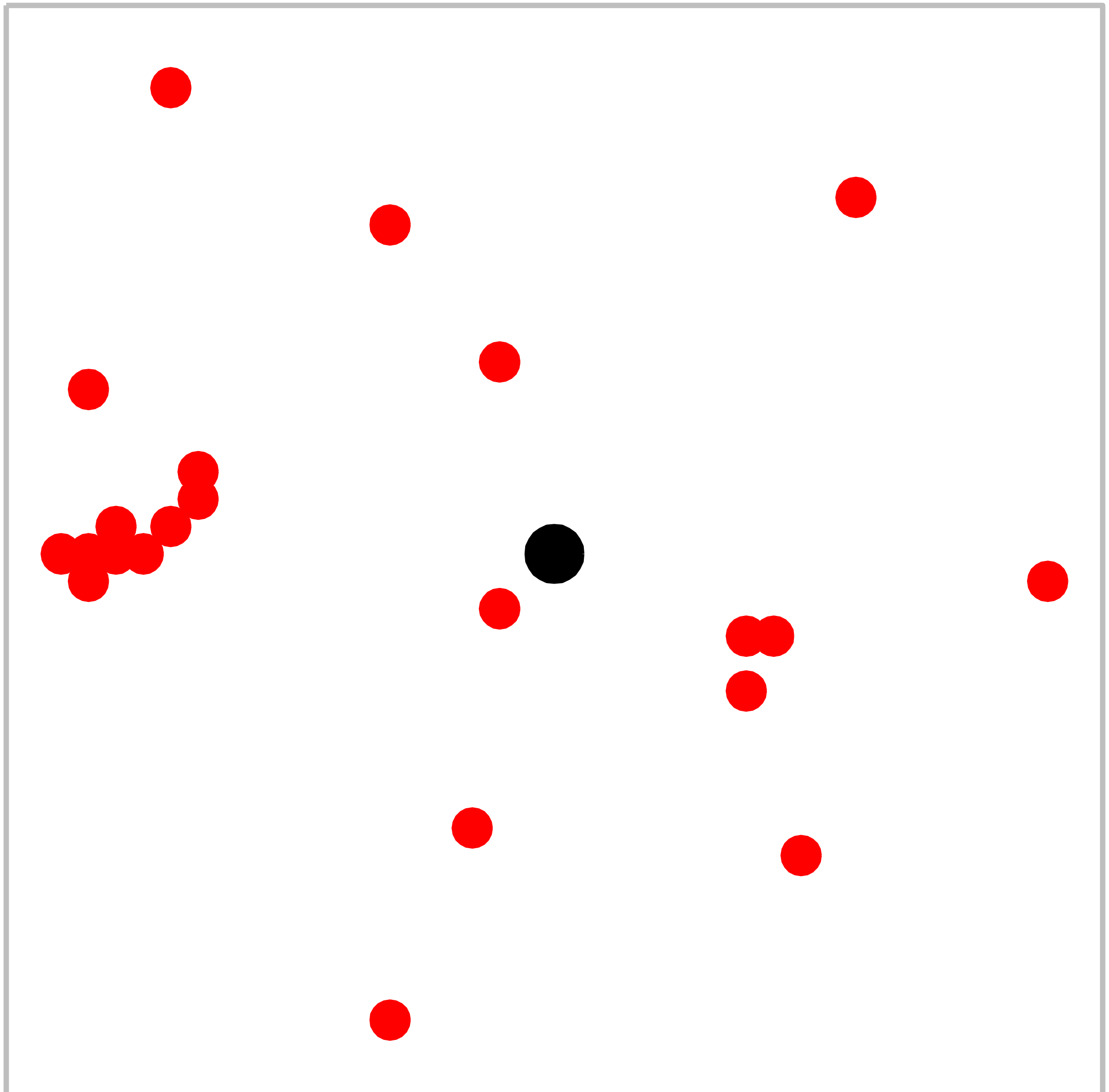
Encapsulation in two-type FPP

Two type encapsulation (Haggstrom-Pemantle)

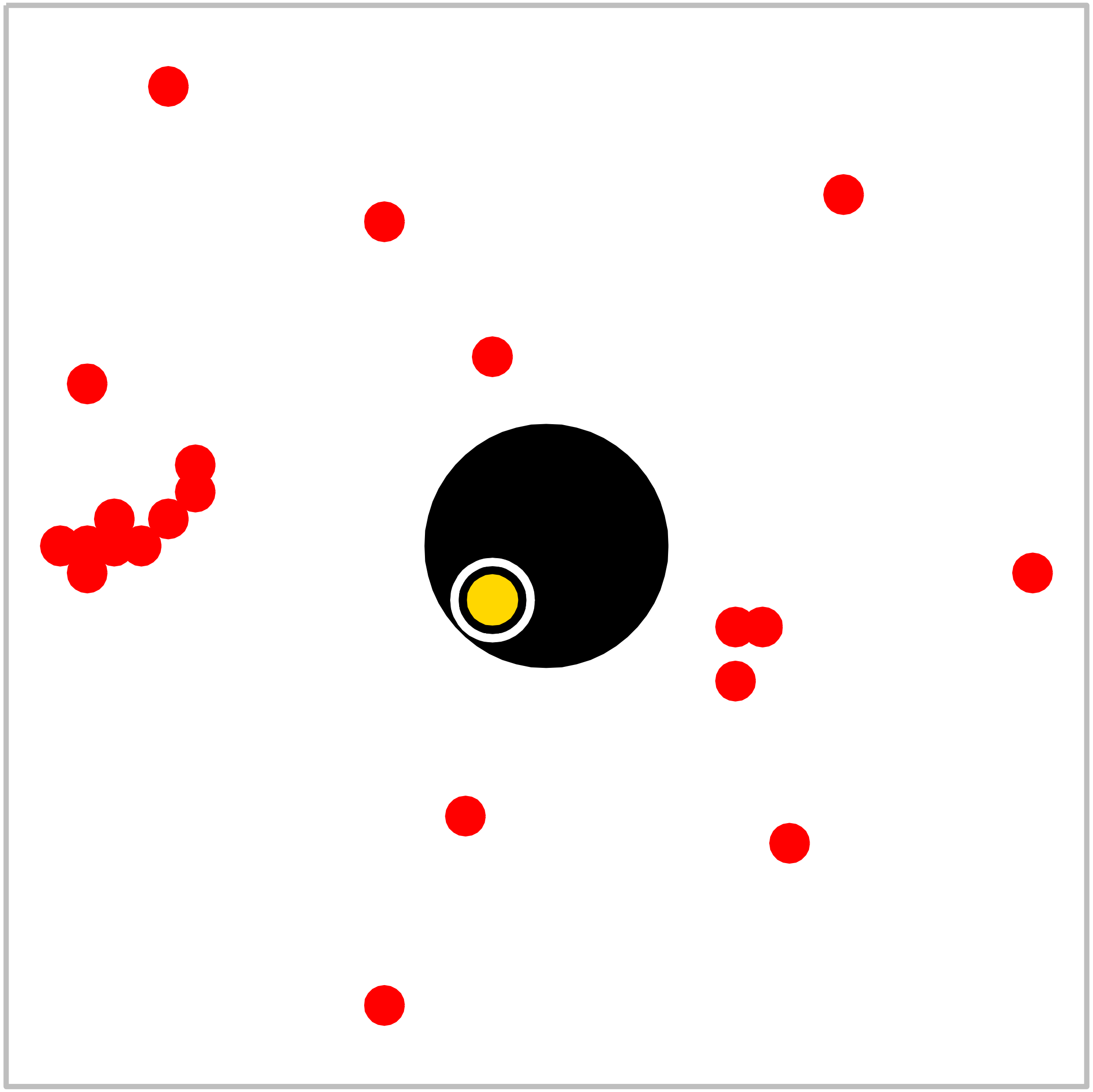


$$\mathbb{P}(\textbf{Type 1} \text{ surrounds } \textbf{Type 2}) \to 1 \quad as \quad \text{dist}(\textbf{type 1}, \textbf{type 2}) \to \infty$$

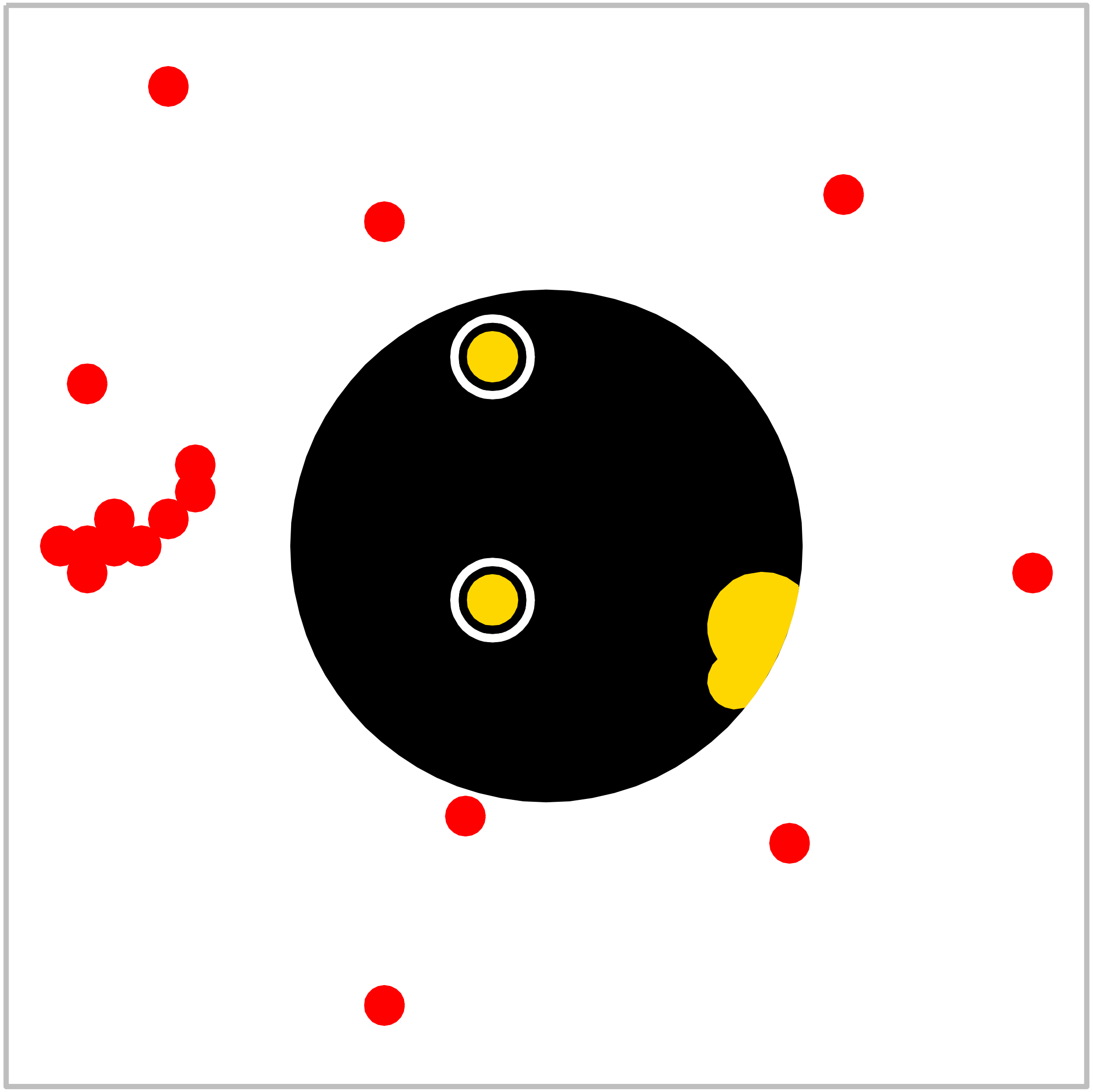
Multi-scale encapsulation



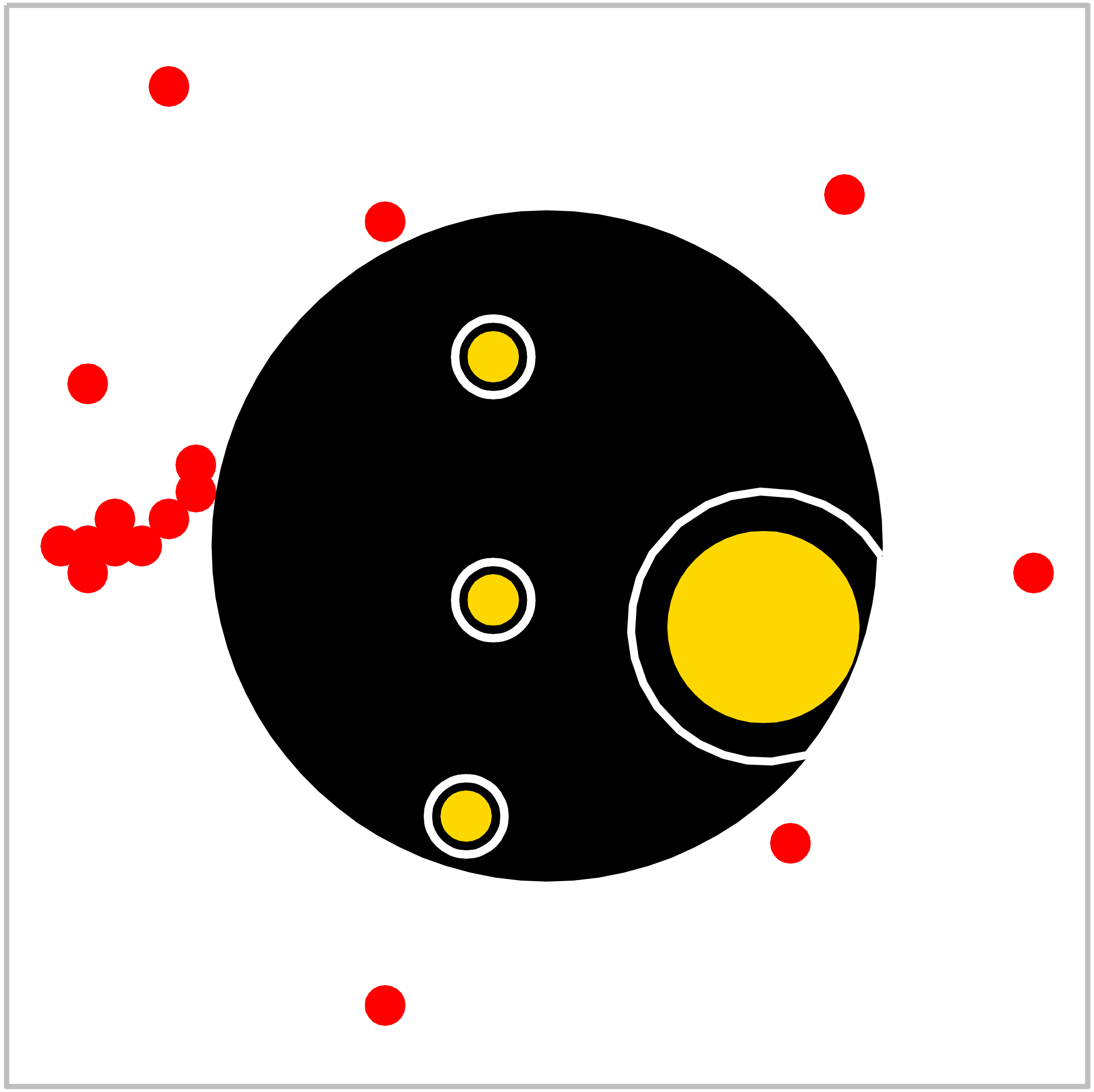
Multi-scale encapsulation



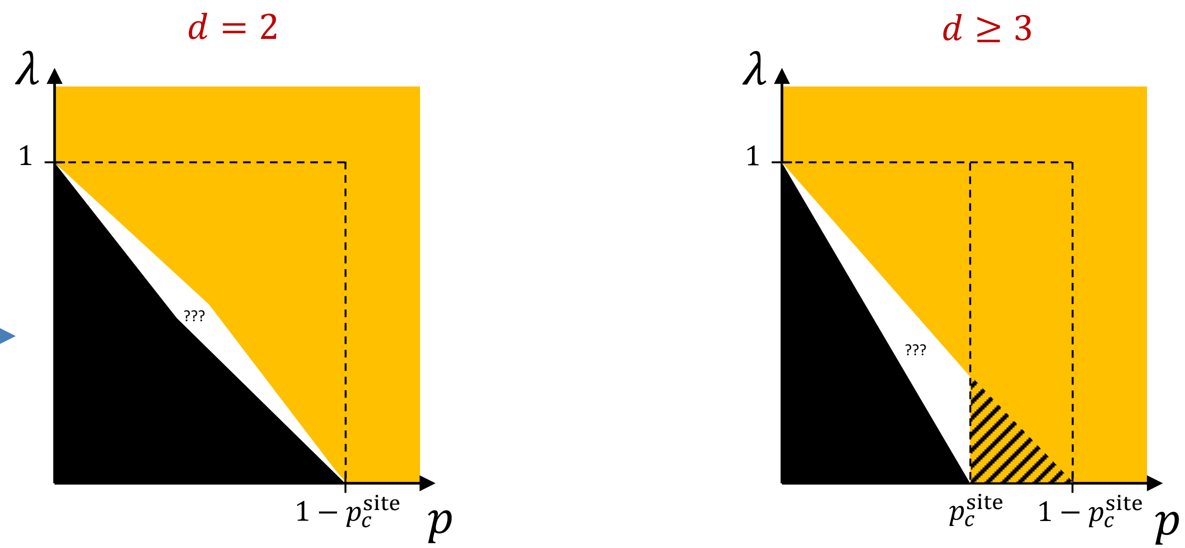
Multi-scale encapsulation



Multi-scale encapsulation



Second result



Sidoravicius, S. →

Theorem [Survival of type 1 for small λ]

For any $p \in (0, 1 - p_c^{\text{site}})$, there exists $\lambda_0 > 0$ such that $\forall \lambda < \lambda_0$
$\mathbb{P}(\textbf{Type 1} \text{ survives}) > 0$

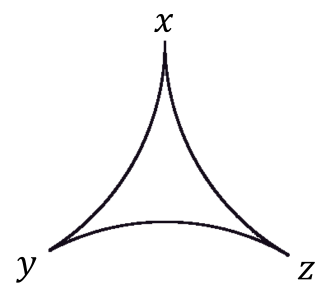
T. Finn and A.S., Coexistence in competing first passage percolation in $d \geq 3$, in preparation

For $d \geq 3$ we have $\left(p_c^{\text{site}}, 1 - p_c^{\text{site}}\right) \neq \emptyset$

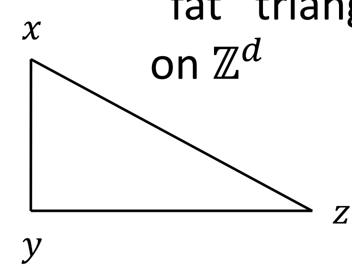
Hyperbolic (and nonamenable) graphs

Hyperbolic graphs

- All triangles are δ-thin



vs



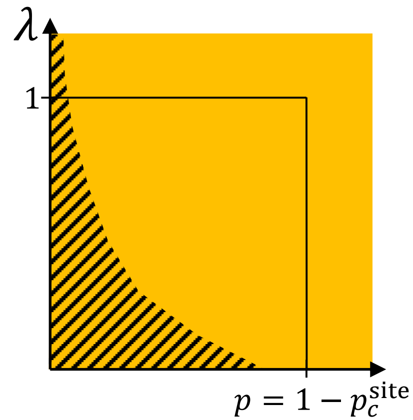
"fat" triangles on $\mathbb{Z}^d$

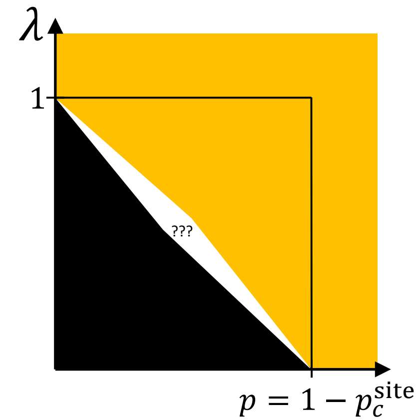
Theorem [Type 2 survives]

For any $\lambda > 0$, any $p > 0$

$$\mathbb{P}(\textbf{Type 2} \text{ survives}) = 1$$

E. Candellero and A.S. Coexistence of competing first passage percolation on hyperbolic graphs, *submitted*





Theorem [Coexistence]

For any $\lambda > 0$, there is $p_0 > 0$ s.t. $\forall p < p_0$

$$\mathbb{P}(\textbf{Type 1} \text{ survives}) > 0$$

Coexistence: overall picture

Coexistence is known to hold in the following cases:

- Hyperbolic, non-amenable graphs (Candellero, S.)

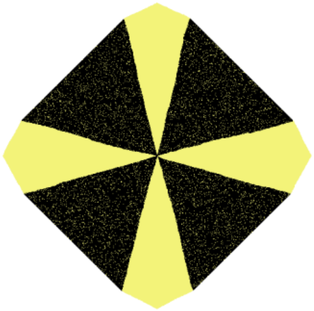
  Type 2 always survive

- $\mathbb{Z}^d, d \geq 3$ (Finn, S.)

  Type 2 always survive

- $\mathbb{Z}^d, d \geq 2$ for deterministic passage times (Sidoravicius, S.)

  Type 1 survive with same speed for all $\lambda < 1$



There is no proof of coexistence when both types have to «fight» to survive