A TSP Objective Function that Ensures Feasibility at Stable Points

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1. Introduction

The convergence of a TSP Hopfield-Tank network to a state representing an optimal tour depends essentially on two main factors. First, there is the question of whether a feasible point has been reached when the network stabilizes, i.e., whether the neuron configurations represent a TSP tour. Secondly, there is the question of how well an optimal TSP tour has been approximated.

Since the original Hopfield-Tank proposal in 1985 (Hopfield and Tank 1985; Hopfield and Tank 1986), much controversy has gone on concerning the validity and efficacy of their model for TSP. Despite much further criticism (Wilson and Pawley 1988) and many improvement proposals (Akiyama et al. 1989; Takefuji and Szu 1989; Van den Bout and Miller 1988; Van den Bout and Miller 1989), the two questions raised above have remained essentially unanswered, except for the incorporation by some researchers of noise terms in the TSP formulation to help escape from local minima. Although the issue of trying to ensure the feasibility of stable points has also been touched, the resulting proposals invariably end up in formulations that depend crucially on the heuristic choice of a set of constants whose determination carries absolutely no rigor.

In this paper we show how a TSP objective function can be constructed in such a way that the feasibility of all stable points of the network's state space is ensured. The main point of our proposal is that the neurons' threshold potentials too can be used in coding a particular TSP instance, instead of synaptic weights only. We then prove that, given certain relations among these parameters, every stable state corresponds to a feasible TSP tour, and that, in particular, the shortest TSP tour must be stable.

Once the feasibility of all stable points is guaranteed, one may then concentrate on the issue of embedding some hill-climbing behavior into the functioning of the network. We do so by utilizing an analogy with the functioning of the Boltzmann Machine (Hinton et al. 1984), which is essentially a Hopfield binary-neuron network (Hopfield 1982) where neurons are occasionally allowed to disobey the threshold rule that determines a neuron's state given its input potential. Likewise, when continuous-response neurons are used, at each infinitesimal time step of the network's evolution we give all neurons a chance to disobey their input-output sigmoidal characteristic, employing Simulated Annealing (Kirkpatrick et al. 1983) for control.

Our experimental results are promising — of all trials on a same TSP instance starting at randomly generated initial configurations, nearly half yielded optimal solutions.

We describe our TSP formulation in Section 2, and our improved neural model (along with experiments) in Section 3.

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2. TSP Formulation

In the continuous-neuron Hopfield model (Hopfield 1984), the state $v_i$ of a neuron $N_i$ may assume any value between the two extremes 0 and 1. The input potential $u_i$ to $N_i$ follows the differential equation

$$C_i \frac{du_i}{dt} = \sum_{j \neq i} w_{ij}v_j - \frac{u_i}{R_i},$$

where $w_{ij}$ is the synaptic weight from $N_j$ to $N_i$, $C_i$ is the neuron's capacitance, and $R_i$ its resistance. The values of $v_i$ and $u_i$ are interrelated by the sigmoidal relationship

$$v_i = f_i(u_i) = \left[1 + e^{-\gamma_i(u_i - \theta_i)}\right]^{-1},$$

where $\gamma_i > 0$ is the sigmoid's "gain," i.e., it regulates the steepness of the response curve around the threshold point $u_i = \theta_i$.

For symmetric weights $w_{ij} = w_{ji}$, the time-derivative of an energy function $E$ is strictly negative, indicating that the evolution of the network in time promotes the search for a local minimum of that energy, which for high gain situations is approximately given by

$$E = -\frac{1}{2} \sum_{i \neq j} w_{ij}v_i v_j + \sum_i \frac{\theta_i}{R_i} v_i.$$

In TSP we are given $n$ cities and the distances $d_{AB}$ between any two cities $A$ and $B$. The problem then asks for a shortest tour that visits every city exactly once. In the Hopfield-Tank model (Hopfield and Tank 1985; Hopfield and Tank 1986), TSP is formulated with the aid of $n^2$ continuous-response neurons organized in a matrix. Each row in the matrix of neurons corresponds to a city, while each column corresponds to one of the possible positions a city may occupy in a tour. A stable state of such a network is a feasible stable state if it comprises exactly $n$ neurons in the on state (close to 1), arranged in the matrix so that there is one of them per row and one of them per column.

Our formulation of TSP follows that of the Hopfield-Tank model, except that we utilize the neurons' thresholds, in addition to the synaptic weights, to program the particular instance of TSP at hand. In connection with our TSP formulation and experiments, we assume that, for all neurons $N_i$, $R_i = R$, $C_i = C$, $\theta_i = \theta$, and $\gamma_i = \gamma$.

Let $N_i$ and $N_j$ be two neurons in the matrix. If they occupy the same row or the same column in the matrix, then they are interconnected by the inhibitory synapse $w_{ij} = W < 0$, in order to ensure that there will not be more than one neuron in the on state per row or per column. If the neurons $N_i$ and $N_j$ are in rows corresponding to cities $A$ and $B$, respectively, and furthermore they occupy adjacent columns (notice that columns 1 and $n$ are adjacent to each other), then they are interconnected by the inhibitory synapse of weight $w_{ij} = -d_{AB}$. All other synapses in the network receive value zero. In order to make sure that no less than $n$ neurons will be in the on state in a stable state, we provide $\theta$ with a negative value as well.

Proposition 1. If

$$W < \frac{\theta}{R} < \min_A \left\{ -2 \sum_{B \neq A} d_{AB} \right\},$$

then every stable state is feasible.

Proof. We show that every infeasible state is unstable. For consider an infeasible state of the network; one of the following three cases must happen: (a) there are $n$ neurons in the on state, but not one per row and one per column; (b) there are more than $n$ neurons in the on state; (c) there are less than $n$ neurons in the on state.
If case (a) or case (b) holds, then at least one row (say $k$) and one column (say $l$) are such that row $k$ has $x \geq 1$ neurons in the on state and column $l$ has $y \geq 1$ neurons in the on state, in such a way that $x + y > 2$ and that the neuron (say $N_i$) occupying the position $(k, l)$ in the matrix is in the on state. Because $N_i$ is in the on state, we have $u_i > \theta$, and then the time derivative of $u_i$ becomes

$$C \frac{du_i}{dt} < (x + y - 2)W - \frac{\theta}{R},$$

where we have taken into account the strict negativity of the synaptic weights that connect $N_i$ outside its row or column. By the first inequality in the hypothesis, and considering that $x + y > 2$, it then follows that

$$\frac{du_i}{dt} < 0.$$

Similarly, if case (c) holds, then there must exist one row $k$ and one column $l$ with no neuron in the on state at all. If $N_i$ is the neuron in position $(k, l)$ in the matrix, then we have $u_i < \theta$, and so

$$C \frac{du_i}{dt} > -2 \sum_{B \neq K} d_{BK} - \frac{\theta}{R},$$

where we have taken $K$ to be the city corresponding to row $k$. By the second inequality in the hypothesis we get

$$\frac{du_i}{dt} > 0.$$

So either we have a neuron $N_i$ for which $u_i > \theta$ and $du_i/dt < 0$, or a neuron $N_i$ for which $u_i < \theta$ and $du_i/dt > 0$. As a consequence, every infeasible state is unstable. ■

By Proposition 1, it follows that the energy of the neural network proposed in this section is, at stable points, equal to the length of some tour plus the constant $n\theta/R$. Also, the optimal tour is, by Proposition 1, necessarily stable. Consequently, the problem of finding a global minimum of the energy function $E$ is equivalent to TSP.

3. An Enhanced Neural Model and Simulation Results

We describe our hill-climbing mechanism for use with the Hopfield-Tank model in terms of a discrete-time simulation algorithm to integrate the differential equations describing the network's evolution. In this simulation algorithm, at each time step (or iteration) the input potential $u_i$ of every neuron is evaluated, and then its new state is computed via the assignment $v_i := f_i(u_i)$. In order to allow occasional energy increases, we will sometimes let $N_i$ disobey its input-output characteristic. This will be done at the end of iterations, so that instead of deterministically performing the assignment $v_i := f_i(u_i)$, we will sometimes do $v_i := 1 - f_i(u_i)$, which corresponds to "flipping" $N_i$'s sigmoid response curve around the potential $u_i = \theta_i$, following the analogy with the Boltzmann Machine. The intuition behind this proposal can be somewhat strengthened by the following observation. The time derivative of the energy $E$ (Hopfield 1984),

$$\frac{dE}{dt} = - \sum_i C_i \left( \frac{dv_i}{dt} \right)^2 f_i^{-1'}(v_i),$$

can only be made positive if at least one of the sigmoid is "flipped" (it is certainly made positive if all sigmoids are "flipped").

The algorithm we propose makes use of a temperature-like parameter $T(t)$ that is continuously decreased and is meant to regulate the acceptance of moves in which $E$ increases, in a Simulated Annealing fashion. At each iteration two sets of neuron states are generated, one in which the direct sigmoids are used, and
another in which the “flipped” sigmoid is used. Then the energy gap $\Delta E$ is computed between the two sets of values, of energy $E_1$ and $E_2$, respectively. The gap

$$\Delta E = E_1 - E_2$$

$$= -\frac{1}{2} \sum_{i \neq j} w_{ij}(v_i + v_j - 1) + \sum_i \frac{\theta_i}{R_i}(2v_i - 1)$$

is then used to probabilistically choose between the two sets. The second set, the one in which the sigmoid were “flipped” to yield the neuron states, is chosen with probability

$$\frac{1}{1 + e^{-\Delta E/T(t)}}$$

otherwise the first set is picked, and the iteration terminates.

From our previous discussion on the time derivative of $E$, it is clear that the energy gap $\Delta E$ is always nonpositive, so along the cooling process the probability of accepting uphill moves varies from its initial value of approximately 0.5 (given a sufficiently high $T(0)$) to a final value of approximately 0. This means that as the system freezes through the decrease of $T(t)$ the set of neuron states generated with “flipped” sigmoid is less and less often chosen, and eventually the system stabilizes as in the deterministic Hopfield-Tank network.

For our simulations on TSP we have employed the temperature decrease function

$$T(t) = T_0 e^{-t},$$

where $0 < \alpha < 1$, $t \geq 0$, and $T_0$ is $O(-n^3 W)$ (so that the probability of accepting an uphill move be approximately equal to 0.5 at $t = 0$). Readily, this function implies that the time $t$ to reach a final temperature $T_f$ (beyond which uphill moves are no longer accepted and the system evolves deterministically) from $T_0$ is given by

$$t = \log_{1/\alpha} \left( \frac{T_0}{T_f} \right)$$

$$= O\left(\log_2 n + \log_2(-W)\right),$$

and therefore grows relatively slowly with the size of the problem (notice that, by Proposition 1, the value of $W$ depends on the distances between cities). This time $t$ can also be taken as the time for final quiescence, since the time elapsed from the instant at which $T_f$ is reached to the instant of final stabilization can be taken as approximately constant.

Our tests have been performed with $R = C = 1$, $\gamma = 1$, $\Delta t = 0.001$, and $T_f = 1$, and on instances of size $n = 6, 8, 10, 12$ with values for $W$ and $\theta$ conforming to Proposition 1. The corresponding temperature decrease rate was $\alpha = 0.6$, and initial neuron states were generated randomly. Of all tours obtained for a same instance of TSP, nearly 50% have been optimal, thereby characterizing a very satisfactory performance, especially if we consider the relatively modest time it takes for stabilization.

References


